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TWO MATHEMATICAL ESSAYS:

THE FIRST

ON ULTIMATE RATIOS,

THE SECOND

ON THE POWER OF THE WEDGE.

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The ſubſtance of the following Eſſays was part of a courſe of lectures delivered in St. John's College, Cambridge: but having been imperfectly taken down by ſome of the hearers, the author thought proper to reduce it into writing himſelf, and has been prevailed upon to make it public.

ESSAY I.

On ULTIMATE RATIOS.

1. A Great part of the labour of geometricians has been employed in comparing the areas of curvilinear figures with rectilinear figures, and the areas of curvilinear figures with each other; not merely on account of the importance of the ſubject itſelf, but becauſe almoſt all queſtions in philoſophy can be reduced to a compariſon of ſuch areas. For inſtance; there are no two forces in mechanics however related to each other, but geometricians can feign two figures ſo circumſtanced that their areas ſhall have the ſame proportion to each other as the propoſed forces; therefore any theorems ſerving to diſcover the unknown proportion of thoſe geometrical figures from the aſſumed circumſtances, will diſcover likewiſe the proportion of the quantities analogous to them. An example of this

we have in the doctrine of falling bodies, in which the proportion of the right lines deſcribed in their deſcent when the times of their fall are different, is found by comparing the areas of two right angled triangles. For if the perpendicular heights of thoſe triangles be aſſumed proportional to the reſpective times of deſcent and their baſes to the velocities acquired in the fall, it is demonſtrable that the right line deſcribed in their deſcent will be proportional to the areas of thoſe triangles; and thus the right line deſcribed by heavy bodies in their fall, is ſaid to be *repreſented* by the area of a right angled triangle.*

2. We find Newton frequently prefacing his problems with this expreſſion, *Conceſſis figurarum curvilinearum quadraturis, requiritur, &c.* that is by feigning ſome geometrical figure whoſe area he will ſhow to be analogous to the quantity ſought, the problem is reduced to a geometrical one. By this means we are aſſured that the conditions of the problem involve no inconſiſtency but that it is poſſible; and if the art of comparing curvilinear figures be ſuffici-

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* *Keil's* Phyſics Lect. XI. Theorem XVII. *Cotes De deſcenſu gravium.* *Mac Laurin's* Account, &c. B. II. Chap. 5.

ently underſtood, the ſolution itſelf may be drawn from hence, and perhaps many other circumſtances may be inferred, which otherwiſe would not eaſily appear.

3. The primary method of comparing the magnitude of rectilinear figures, is by laying them upon one another, it being an axiom that ſuch as agree in all their parts are equal.* Thus all the right lined figures which in the firſt book of the *Elements* are proved to be equal, are always ſhown to be the ſum or difference of ſuch as would reſpectively cover each other. This is the direct and beſt way of reaſoning, in which the agreement of the extreme ideas is ſhown by their continual agreement with intermediate ideas of the ſame kind: but this method cannot be applied in comparing curvilinear figures with rectilinear ones, becauſe no part of a curve line, can ever coincide with a ſtraight line.

4. To extend geometry to the comparing curvilinear with rectilinear figures the antients made uſe of the method of *Exhauſtions*; † an example of which we have in the ſecond propoſition of the twelfth book

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* Axiom 8 and prop. 4. El. 1.

† See *Archimedes de dimenſione circuli, &c.*

of Euclid. I ſhall only obſerve upon it, that the argument made uſe of is what is called *reductio ad abſurdum*, which though ſtrictly logical is always tedious, inaſmuch as every propoſition muſt be divided into two caſes, in one of which you are to ſhow that the former of the quantities to be compared together is not greater than the latter, and in the other that it is not leſs.

5. To contract this method of reaſoning, *Cavalerius* propoſed what is called the method of *Indiviſibles*,* in which he was followed by Dr. *Wallis* and others of the laſt century. In this method every line is ſuppoſed to conſiſt of a number of other lines of the *ſmalleſt poſſible* length; every curve was conſidered as a polygon each of whoſe ſides is one of thoſe *indiviſible* lines; with other like ſuppoſitions equally abſurd and ungeometrical. Theſe principles ſoon led their followers into perplexity and ofttimes into error, nor was it eaſy to fix bounds to thoſe liberties when once introduced.

6. To avoid both the tedioufneſs of the antients and the inaccuracy of the moderns, Sir Iſaac Newton introduced what he called

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* See *Geometria indiviſibilibus promota.* Ed. 1635.

the method of *prime and ultimate ratios*, the foundation of which is contained in the firſt Lemma of the firſt book of the *Principia*. Many difficulties have been ſtarted and much controverſy raiſed concerning the proof of it; * all which would have been avoided had either the author or his readers obſerved that he is in reality laying down the *definition* of a term, (namely, being *Ultimately equal*,) and not *proving a propoſition*. Taking therefore this firſt Lemma for a definition it may be explained, (not *proved*) in the manner following.

7. Let there be two quantities, one fixed and the other varying, ſo related to each other that *firſt* the varying quantity continually approaches to the fixed quantity. *Secondly*, that the varying quantity does never reach or paſs beyond that which is fixed. *Thirdly*, that the varying quantity approaches nearer to the fixed quantity than by any aſſigned difference, † then is ſuch a fixed quantity called the *limit* of the varying quantity; or in a looſer way of ſpeaking, the varying quantity may be ſaid to be *ultimately*

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* See ſeveral pamphlets written by Mr. *Robins* and Dr. *Jurin* in the years 1736 and 1740, and other papers in the memoirs of literature for thoſe and the intermediate years.

† *Proprius accedunt quam pro* data *quavis differentia*.

timately equal to the fixed quantity.[*] Theſe three properties may be otherwiſe expreſſed more diſtinctly thus. *Firſt*, the difference between the varying quantity and the fixed quantity muſt continually decreaſe. *Secondly*, This difference muſt never become either nothing or negative. *Thirdly*, This difference muſt become leſs in reſpect to the fixed quantity than by any aſſigned ratio; or the difference between the two quantities muſt become a leſs part of the fixed quantity than any fractional part that is aſſigned, how ſmall ſo ever the fraction expreſſing ſuch part may be.[†] Wherever theſe properties are found, the fixed quantity is called the *limit* of the varying quantity, or the varying quantity is *ſaid* to be *ultimately equal* to the fixed quantity. This laſt phraſe muſt not be taken in an abſolute literal ſenſe, there being no *ultimate ſtate*, no particular magnitude that is the *ultimate magnitude*

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[*] Some writers ſeem to have dropped the ſecond condition out of their idea of a limit; however Newton plainly ſuppoſes it. His words are —— *Ultimæ rationes* —— *revera non ſunt rationes quantitatum ultimarum, ſed limites ad quos quantitatum ſine limite decreſcentium rationes ſemper appropinquant*; *et quas proprius aſſequi poſſunt quam pro data quavis differentia,* nunquam vero tranſgredi, *&c.*
Scholium ad Lemma XI. *Lib.* I. *Princip. pag.* 38. *Ed.* 3.

[†] See the definition of proportion given in *Saunderſon*'s Algebra art. 15. page 76. Ed. quarto. or p. 89. Ed. octavo.

nitude of ſuch a varying quantity. Under the word quantity in this definition muſt be included not only numbers, lines, &c. but more eſpecially ratios conſidered as a peculiar ſpecies of quantity; but as the conſideration of ratios which have limits is difficult, we ſhall begin with examples of other quantities.

EXAMPLES IN NUMBERS.

8. Let there be formed a ſeries whoſe firſt term is unity, ſecond $\frac{1}{2}$, third $\frac{1}{4}$, fourth $\frac{1}{8}$, and ſo on, every term being half the preceding term thus; $1 . \frac{1}{2} . \frac{1}{4} . \frac{1}{8} . \frac{1}{16}$ &c. and let the ſum of an indefinite number of terms in this ſeries be conſidered as continually varying, being continually increaſed by the acceſſion of a new term; thus the ſum of two terms is $1\frac{1}{2}$, of three terms is $1\frac{3}{4}$, of four terms is $1\frac{7}{8}$, &c. I ſay then that the varying ſum of the terms of this ſeries continually approximates to the fixed number 2 as its limit. For the difference between the ſum of one, two, three, four, &c. terms and the number 2 will be the

numbers $1 . \frac{1}{2} . \frac{1}{4} . \frac{1}{8} . \frac{1}{16}$, &c. ſucceſſively in infinitum. Here it is evident that the terms of this laſt ſeries which expreſs the ſucceſſive differences between the growing ſum of the terms of the former ſeries and the number 2, *firſt* continually decreaſe; *ſecondly* no term in this ſeries of differences can become either nothing or negative; *thirdly* we may continue this ſeries of ſucceſſive differences, 'till we come to a term which ſhall be a leſs part of the fixed number 2, than any fractional part that can be aſſigned, or ſo that this difference ſhall be leſs when compared with the number 2, than in any proportion aſſigned. For inſtance we may continue this ſeries of differences, 'till we come to a term which ſhall be leſs than a $\frac{1}{1000}$ th part of the number 2, or leſs than the $\frac{1}{10\,000}$ th part of the number 2, or 'till you come to a term which ſhall be leſs than any other fractional part of the number 2 which you pleaſe to point out and aſſign. The number 2 then having the conditions laid down in the definition is to be *called* the limit of the ſum of the terms of this infinite ſeries $1 . \frac{1}{2} . \frac{1}{4} . \frac{1}{8} . \frac{1}{16}$ &c. Any infinite ſeries being propoſed, if a number can be pointed out having the conditions before mentioned then ſuch ſeries is ſaid to have a limit.

In the uſual phraſe of mathematicians, the number 2 in our example would be called the ſum of *all* the terms in the propoſed ſeries continued *in infinitum*; but in this caſe the words muſt not be underſtood in their literal ſenſe, it being abſurd to ſpeak of the ſum of *all* the terms of *an infinite* ſeries, that is, of a ſeries which has *no all* to its number of terms. When mathematicians talk of finding the ſum of an infinite ſeries, they neither mean the ſum of any particular number of terms, nor yet the ſum of all the terms, but the *limit* of the ſum of the terms as before explained.

9. No number leſs than 2, for inſtance $1\frac{7}{8}$ can be taken for the limit; for in this caſe it will not anſwer the ſecond condition. In this example the ſum of four terms of the ſeries equals $1\frac{7}{8}$, and the ſum of five terms exceeds it; therefore the difference between this ſum and the number $1\frac{7}{8}$. propoſed as a limit is in the former caſe

nothing, and in the latter negative. No number greater than 2, for inſtance 3, can be taken for the limit, becauſe the laſt condition will be wanting. For if the ſum of any aſſigned number of terms be always leſs than 2, that ſum muſt always want more than unity of the number 3, and conſequently cannot approach nearer to 3 than by the aſſigned difference 1.

10. In like manner the ſum of the ſeries $1 . \frac{1}{3} . \frac{1}{9} . \frac{1}{27}$, &c. continued in infinitum will be $1\frac{1}{2}$; the ſeries of ſucceſſive differences being $\frac{1}{2} . \frac{1}{6} . \frac{1}{18} . \frac{1}{54}$, &c. in infinitum.

11. To give a ſimilar inſtance in Geometry; Let *AI* (in fig. 1.) be an indefinite right line, on which ſet off the equal parts *AB* . *BC* . *CD* . *DE*, &c. on *AB* make a ſquare, on *BC* a parallelogram whoſe breadth *Bb* is half the ſide *AB*, on *CD* a parallelogram whoſe breadth *Cc* is half the breadth of the former parallelogram, on *DE* another whoſe breadth is half *Cc* and ſo on. It is evident that the right lined figure compoſed of all theſe parallelograms though infinite in extent from *A* towards *I*,

is yet finite in area, and is equal to twice the ſquare upon *AB*; that is, though there is no limit to the extent of ſuch a figure from *A* towards *I*, yet there is a limit to its area in our ſenſe of the word limit; nor is it ſtrange that quantities which have no limits in one reſpect, ſhould yet have limits in another.

12. To make the example in par. 8 general. Let $a : b$ expreſs the common ratio of any ſeries of numbers in continual proportion whoſe firſt term is unity; I ſay then that if a be greater than b, ſuch a ſeries will have a limit, *viz.* $\frac{a}{a-b}$, or in other words the ſum of all the terms of that ſeries continued in infinitum will be $\frac{a}{a-b}$. For the terms of this ſeries will be

$$1 \,.\, \frac{b}{a} \,.\, \frac{b^2}{a^2} \,.\, \frac{b^3}{a^3} \,.\, \frac{b^4}{a^4}, \text{ \&c. whence}$$

The ſum of 1 term is 1

2 terms is . . . $\frac{a+b}{a}$

3 terms is $\frac{aa+ab+bb}{aa}$

4 terms is $\frac{a^3+aab+abb+b^3}{a^3}$

Let each of theſe ſums be ſubtracted from the limit $\frac{a}{a-b}$, and we have the ſucceſſive differences

$$\frac{b}{a-b}\cdot\ \frac{b}{a-b}\times\frac{b}{a}\cdot\ \frac{b}{a-b}\times\frac{b^2}{a^2}\cdot\ \frac{b}{a-b}\times\frac{b^3}{a^3}\cdot$$

&c. or if n be any aſſigned number of terms, the difference between the ſum of that number of terms and $\frac{a}{a-b}$ will be $\frac{b}{a-b}\times\left.\overline{\frac{b}{a}}\right|^{n-1}$ * Whence we may obſerve *firſt* that as the number of terms to be ſummed up increaſes, this difference $\frac{b}{a-b}\times\left.\overline{\frac{b}{a}}\right|^{n-1}$ continually decreaſes, becauſe

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* The ſum of any particular number of terms n may be found thus. Put $x=\frac{b}{a}$ and the ſeries will ſtand thus,

$1+x+x^2+x^3+x^4\ \ldots\ldots\ x^{n-1}$ which put $=s$

whence multiplying by x

$x+x^2+x^3+x^4\ \ldots\ldots\ x^{n-1}+x^n=+sx$

ſubtract the latter ſtep from the former and

$1\ *\ *\ *\ *\ \ldots\ldots\ *\ -x^n=s-sx=\overline{1-x}\times s$

whence $s=\frac{1-x^n}{1-x}$ or reſtoring x, $s=\frac{a^n-b^n}{a^{n-1}\times\overline{a-b}}$.

becauſe x is a fraction leſs than unity, its powers will continually decreaſe; by increaſing n the number x^n may become leſs than any aſſigned fraction, that is, when n is infinite x^n vaniſhes and we have $s=\frac{1}{1-x}$ or (reſtoring x) $s=\frac{1}{1-\frac{b}{a}}=\frac{a}{a-b}$ as before.

cauſe $\frac{b}{a}$ being a fraction leſs than unity, its powers continually decreaſe. *Secondly*, this difference can never become nothing or negative; the powers of a fraction though they decreaſe being always real and affirmative. *Thirdly*, this difference may become leſs in reſpect of $\frac{a}{a-b}$ than by any aſſigned proportion. For $\frac{a}{a-b}$ is to $\frac{b}{a-b} \times \left.\overline{\frac{b}{a}}\right|^{n-1}$ as a^n is to b^n or as $\left.\overline{\frac{a}{b}}\right|^{n}$ is to 1; but $\frac{a}{b}$ is greater than unity, therefore the powers of $\frac{a}{b}$ continually increaſe, and by increaſing the index n the number $\left.\overline{\frac{a}{b}}\right|^{n}$ may become greater than any aſſigned number, conſequently $\left.\overline{\frac{a}{b}}\right|^{n}$ may become greater in reſpect of 1, or 1 leſs in reſpect of $\left.\overline{\frac{a}{b}}\right|^{n}$ than by any aſſigned ratio; therefore the difference $\frac{b}{a-b} \times \left.\overline{\frac{b}{a}}\right|^{n-1}$ may become a leſs part of $\frac{a}{a-b}$ than any aſſigned fractional part. The quantity $\frac{a}{a-b}$ therefore having theſe three properties agrees with the definition of a limit of the growing ſum of the terms of the ſeries, or we may ſay (in the language of mathematicians) that it is equal to the ſum of all the terms continued in infinitum.

13. What has been thus proved may be ſhown more conciſely by dividing a by $a-b$ in the manner of compound diviſion in arithmetic;[*] for the quotient will be the very ſeries propoſed in the example, and it may be further confirmed by multiplying that ſeries by $a-b$, for the product will be a, all the terms except the firſt deſtroying one another; therefore a divided by $a-b$ is equal to that whole ſeries.[†]

14. We may obſerve that if every term of the forcgoing ſeries be multiplied by any

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[*] See Saunderſon's Algebra, page 73. Quarto Edition.

[†] Hence again we may find the ſum of n terms in this ſeries. For putting $x = \frac{b}{a}$ as before, the laſt or n^{th} term will be x^{n-1}, and the remaining terms of the ſeries will be $x^n + x^{n+1} + x^{n+2} + x^{n+3}$, &c. $= x^n \times \overline{1 + x + x^2 + x^3 + x^4, \text{\&c.}} = x^n \times \frac{1}{1-x} = \frac{x^n}{1-x}$; Subtract this from $\frac{1}{1-x}$ or the ſum of all the terms, and there will remain the ſum of the firſt n terms $= \frac{1-x^n}{1-x}$ as before.

number c it will become $c + \frac{bc}{a} + \frac{b^2}{a^2} \times c + \frac{b^3}{a^3} \times c$, &c. that is, a ſeries of numbers in continual proportion whoſe common ratio is that of a to b as before, but whoſe firſt term is c. It is evident therefore, that in this caſe, to find the ſum of all the terms we muſt multiply the former limit by c, that is, it will be $\frac{ac}{a-b}$.

15. *Example* 1. Let the firſt term be 1, and the common ratio 10 to 1, then the ſeveral terms of the ſeries are $1 . \frac{1}{10} . \frac{1}{100} . \frac{1}{1000}$, &c. or if we expreſs them decimally

the firſt term is 1,
2d 0,1
3rd 0,01
4th 0,001

their ſum 1,111 &c. $= \frac{10}{10-1} = \frac{10}{9}$

Example 2. Let the firſt term be 1, the common ratio 9 to 1, and the terms expreſſed decimally are

the 1ſt term	1,
2d	0,111111
3d	0,012345
4th	0,001371
5th	0,000152
6th	0,000017

their ſum $1,124996 = \frac{9}{8} = 1,125.$

16. Before we proceed to conſider ratios that have limits we muſt learn to diſtinguiſh between the terms of a ratio and the ratio itſelf. The terms of a ratio may vary through all degrees of magnitude, and yet the ratio itſelf continue invariable. Let AQ (fig. 2.) be an indefinite right line, AI another interſecting it in a given angle at A, and on any point P in AQ draw PM at right angles to AQ meeting AI in M. Let the point P move upon the right line AQ ſo that AP and PM may increaſe and decreaſe. The lines AP and PM in this caſe may vary through all degrees of magnitude, may become greater or leſs than any aſſigned line; that is, in the common language of mathematicians, may become infinitely great or infinitely little, yet the ratio of AP to PM is invariably the ſame, becauſe the triangles APM are always ſimilar.

In this inſtance then, varying the terms does not vary the ratio at all; in others varying the terms in infinitum does likewiſe vary the ratio in infinitum; and there are inſtances where although changing the terms does change the ratio, yet the ratio never varies beyond certain limits, even though the terms themſelves ſhould increaſe or decreaſe in infinitum. The following example of this is borrowed from *Saunderſon*'s algebra, art. 336.

17. Let x be any varying quantity; make $4xx + 3x = A$ and $2xx + x = B$, then will A and B alſo be varying quantities as depending upon x; when x vaniſhes A and B will both vaniſh, and when x is infinite they will both be infinite. I ſay that the ratio of A to B while x decreaſes in infinitum approximates to the ratio of 3 to 1 as its limit. For *firſt* A is to B as $4xx + 3x$ is to $2xx + x$ or as $4x + 3$ is to $2x + 1$. Here it is eaſy to ſee that as x decreaſes the quantities $4x$ and $2x$ decreaſe, and conſequently the ratio of $4x + 3$ to $2x + 1$ or A to B approaches to that of 3 to 1. *Secondly*, the ratio of A to B can never reach or exceed that of 3 to 1. For $6xx + 3x$ is to $2xx + x$ as 3 is to 1, but $4xx + 3x$

is a leſs quantity than $6xx + 3x$, therefore $4xx + 3x$ is to $2xx + x$ or A is to B always in a leſs ratio than that of 3 to 1. *Laſtly*, the ratio of A to B will approach nearer to that of 3 to 1 than by any aſſigned difference. For in the terms of this ratio $4x + 3$ and $2x + 1$, the varying parts $4x$ and $2x$ by diminiſhing x may become leſs than any aſſigned quantity while the other parts 3 and 1 remain the ſame; therefore the ratio of A to B will approach nearer to the ratio of 3 to 1, than by any aſſigned difference.

In like manner the ratio of A to B, while x increaſes in infinitum approximates to the ratio of 2 to 1 as its limit. For ſince A is to B as $4x + 3$ is to $2x + 1$ or as $4 + \frac{3}{x}$ is to $2 + \frac{1}{x}$ it is eaſy to ſee that as x increaſes the quantities $\frac{3}{x}$ and $\frac{1}{x}$ decreaſe and conſequently the ratio of $4 + \frac{3}{x}$ to $2 + \frac{1}{x}$ or A to B approaches to the ratio of 4 to 2 or 2 to 1. Again; the ratio of A to B can never be leſs (that is nearer to the ratio of equality) than the ratio of 2 to 1. For

$4xx + 2x$ is to $2xx + x$ as 2 is to 1, but $4xx + 3x$ is a greater quantity than $4xx + 2x$; therefore $4xx + 3x$ is to $2xx + x$ or A is to B always in a greater ratio than that of 2 to 1. Laſtly; the ratio of A to B will approach nearer to that of 2 to 1, than by any aſſigned difference. For in the terms of this ratio $4 + \frac{3}{x}$ and $2 + \frac{1}{x}$, the variable parts $\frac{3}{x}$ and $\frac{1}{x}$, by increaſing x may become leſs than any aſſigned fraction while the parts 4 and 2 remain the ſame; therefore the ratio of A to B will approach nearer to the ratio of 4 to 2 or 2 to 1 than by any aſſigned difference.

Here then we ſee that though diminiſhing x and conſequently diminiſhing the terms A and B increaſes their ratio, and contrariwiſe increaſing theſe terms by increaſing the quantity x decreaſes their ratio, yet there is a limit both to the increaſe and decreaſe of this ratio, although there is none to the terms themſelves that compoſe it.

18. The ratio of 3 to 1 which limits the ratio of A to B when theſe terms decreaſe in infinitum is called the *ultimate ratio* of the evaneſcent quantities A and B; and it

is likewiſe called, but more rarely, the *firſt ratio* of the naſcent quantities A and B. The ratio of 2 to 1 which is their other limit, is called the *ultimate ratio* of the quantities A and B increaſing in infinitum.

18. Another example of a ſimilar nature we have as follows. Let x be a varying quantity and d a conſtant one, then will $x+d$ and x be two varying quantities capable of all degrees of magnitude. I ſay that the ratio of $x+d$ to x, while x increaſes will continually decreaſe, but not beyond a certain limit, which limit is the ratio of equality. On the contrary, if x decreaſe, the ratio of $x+d$ to x will continually increaſe more and more in infinitum and never come to a limit. For $x+d$ is to x as $1+\frac{d}{x}$ is to 1; now as x increaſes, the fraction $\frac{d}{x}$ decreaſes and may become leſs than any aſſigned fraction; but the number 1 which is the other part of the antecedent of this ratio remains the ſame, ſo likewiſe does the conſequent of this ratio, therefore the ratio of $1+\frac{d}{x}$ to 1 continually approximates to the ratio of equality. *Secondly*;

the fraction $\frac{d}{x}$ has always ſome magnitude, therefore the antecedent $1 + \frac{d}{x}$ is ever greater than the conſequent 1; therefore the ratio of $1 + \frac{d}{x}$ to 1 can never reach the ratio of equality, much leſs paſs it ſo as to become a ratio *minoris inæqualitatis*, or a ratio in which the antecedent is leſs than the conſequent. *Laſtly*, the varying fraction $\frac{d}{x}$, as x increaſes, will become leſs than any aſſigned fraction, while the other part of the antecedent and likewiſe the conſequent of this ratio remains the ſame. Therefore this ratio of $1 + \frac{d}{x}$ to 1 will approach nearer to the ratio of equality than by any ratio that can be aſſigned, how ſmall ſoever ſuch ratio may be; therefore the ratio of equality is the limit of the ratio of $1 + \frac{d}{x}$ to 1, and conſequently the limit of the ratio of $x + d$ to x which continually decreaſes while the terms which compoſe this ratio continually increaſe in infinitum.

If x decreaſes then $\frac{d}{x}$ will increaſe and may become greater than any aſſigned number, how great ſoever that number may be, while the other part of the antecedent and likewiſe the conſequent remain fixed, each being unity; therefore the ratio of $1 + \frac{d}{x}$ to 1, and conſequently the ratio of $x + d$ to x, as x decreaſes, will become greater than any aſſigned ratio whatever and ſo has no limit to its increaſe. Here we have an inſtance of a ratio that has a limit to its decreaſe, but none to its increaſe.

We may obſerve that though the terms of this ratio, *viz.* $x + d$ and x, never get nearer to each other, their conſtant difference being d, yet the ratio of the terms approximates to the ratio of equality; though the terms get no nearer in their difference, yet (if the expreſſion may be allowed) they get nearer in their ratio. The *difference* of the terms and the *ratio* of the terms, are ideas very diſtinct from each other, and in no wiſe ſo connected, but that one may vary while the other is conſtant. In this inſtance the ratio varies, but the difference of the terms is conſtant; on the contrary, the differference between the two lines *A P* and *PM* in fig. 2. (ſee par. 16.) continually varies, but the ratio of thoſe lines is invariable.

19. Although the ratio of equality may be ſtrictly called the *limit of the varying ratio* of the quantities $x + d$ and x, yet the terms of this ratio can never be ſtrictly ſaid to be equal, no not *ultimately equal*, ſince that plainly ſuppoſes an *ultimate ſtate* in which they are equal; no nor equal when they *vaniſh into infinity*, or when they ſtep out of finite exiſtence into infinity. There is no finite quantity next to infinity; no number (for inſtance) which is the next number to infinity, and therefore no ſtep out of finite into infinite.* Nor can we ſay in ſtrictneſs that two infinitely great numbers with a finite difference are equal, it being a propoſition plainly abſurd and contradictory. There is no ſuch thing in nature as any infinitely great number; and it is contradictory to ſay of any two numbers, both that they have a difference, and that they are equal. Whoever conſiders that the idea of infinity is a general or ab-

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* Neither is there any ſtep out of a *ſtate of nothingneſs* into finite exiſtence. There is no fraction ſo ſmall as to be the *very next* fraction to nothing. No fraction can ever be aſſigned ſo ſmall, but another fraction may be aſſigned that is ſmaller.

ſtract idea, that the idea of number is always particular, that infinity is a *property* of number, a *property* of extenſion, &c. not any number, not any extenſion, &c. itſelf, will eaſily ſee that theſe and ſuch like expreſſions can have no *literal* meaning.* As for the metaphorical uſe of them to avoid circumlocution or the introduction of new terms, it may be allowed (when once the literal meaning has been explained) on this as well as numberleſs other occaſions, both in ſcience and common life.

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* By ſaying that number or that extenſion is infinite, we mean only to aſſert the impoſſibility of limiting the increaſe of number or the increaſe of extenſion: we mean to aſſert the abſurdity of fixing upon any particular number how great ſoever, or any particular extenſion how large ſoever, as the biggeſt poſſible number or the greateſt poſſible extenſion. There is no ſuch thing exiſting as a number actually infinite, or an infinite right line. Euclid requires you to allow the poſſibility of producing a right line as far as ever he is pleaſed to direct, or as ſome would ſay, the poſſibility of producing it *in infinitum*, but he makes no propoſitions about infinite right lines —— ἐκβαλλόμεναι αἱ δύο αὗται εὐθεῖαι ἐπ' ἄπειρον —— in the twelfth axiom which is tranſlated by *Commandine*, *rectæ lineæ illæ* in infinitum *productæ*, is rendered by Dr. *Simſon* —— theſe ſtraight lines being continually produced: So likewiſe prop. 12. El. 1. Επὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον, — tranſlated by *Commandine* — *ſuper data recta linea* infinita — is rendered by Dr. *Simſon* —— upon a ſtraight line of an unlimited length. The term *infinite* has been ſo abuſed that it can hardly be admitted in mathematical writings any longer; and it is high time to drop it, when authors talk of adding, ſubtracting, &c. *infinites and infiniteſimals* as familiarly as if they were common numbers.

It may be proper to obſerve here, that as number and extenſion have no limit to their increaſe, ſo neither have they any limit to their decreaſe. It is abſurd to fix upon any particular fraction as the leaſt poſſible number, or to fix on any particular line as the leaſt poſſible extenſion —— There is *no leaſt poſſible* — There exiſts no ſuch thing as a fraction infinitely ſmall, or a line infinitely little; nor did *Euclid* or *Archimedes* or any of the antient geometricians ever ſuppoſe it.

20. When the difference between any two quantities decreaſes ſo as to become a leſs fractional part of one of them than any aſſigned fractional part whatever; or when the difference between the terms of a ratio becomes leſs in reſpect of one of them (the greater for inſtance) than by any aſſigned ratio (according to the third condition required in par. 7) we ſhall expreſs this for the future by ſaying that ſuch difference *vaniſhes* in reſpect of that *(greater)* quantity.

21. Now if the difference between two terms vaniſhes in reſpect of one of them, it will alſo vaniſh in reſpect of the other. It is true that at any aſſigned time, when the terms have a particular magnitude, the difference between the terms will always be a leſs fractional part of the greater term than it is of the leſs term; but as this difference continually decreaſes it will become the ſame fractional part of the leſs term that it was before of the greater term. For inſtance: let the leſſer of the propoſed terms be fixed, but let the greater be variable;

let the leſs term be always 99, but the greater term at a certain point of time be 100; their difference at that particular time is 1, which is a $\frac{1}{99}$th part of the leſs term, but is only an $\frac{1}{100}$th part of the greater term. Now as the difference between the terms is ſuppoſed to continually decreaſe (ſo as to become equal to or leſs than any propoſed fractional part of the greater term) let the difference ſink to 0,99 (ſo as to become an $\frac{1}{101}$th part of the greater term) and it will now be juſt an $\frac{1}{100}$th part of the leſs term; that is, the difference will now be the ſame fractional part of the leſs term that it was before of the greater term. The terms which before were 100 and 99, will now be 99,99 and 99, and their difference which before was 1, will now be 0,99, or an $\frac{1}{100}$th part of 99: thus then if the difference be at any time an $\frac{1}{100}$th part of the greater term, it will (afterwards) alſo be an $\frac{1}{100}$th part of the leſs term; if the difference become a $\frac{1}{1000}$th part of the greater term, it will alſo become a $\frac{1}{1000}$th part of the leſs term; and if the difference between the two quantities vaniſhes in reſpect of one of them, it does alſo vaniſh in reſpect of the other.

22. We may hence enlarge our definition par. 7, and in laying down the third condition ſay, the difference muſt vaniſh in reſpect *either* of the fixed, or of the varying quantity, ſince one of theſe implies the other.

23. What was laid down in par. 21, is true, though neither of the terms be fixed, but both of them increaſe or decreaſe without limit perpetually. In the inſtance before conſidered par. 18, in which the terms were $x + d$ and x, and where x continually increaſing, the terms themſelves do both of them continually increaſe. In this inſtance if the difference vaniſhes in reſpect of one of the terms it will alſo vaniſh in reſpect of the other. Let x be to d at any aſſigned point of time as n is to 1; and d at that inſtant of time will be the $\frac{1}{n+1}$th part of

the greater term, and the $\frac{1}{n}$th part of the leſs term; now though the cotemporary value of theſe fractions can never be equal, yet in ſucceſſion (as x and conſequently n increaſes for ever without limit) the value of the latter fraction will become whatever the former has been. If $\frac{1}{n+1}$ becomes leſs than any aſſigned fraction ſo will $\frac{1}{n}$ likewiſe; and thus if the difference d vaniſhes in reſpect of one of the terms $x+d$ (becauſe x increaſes without limit) ſo will d vaniſh alſo in reſpect of the other term x. The ſame would be true if the terms were x and $x-d$ all things being as before.

24. Again; if x by decreaſing vaniſhes in reſpect of ſome fixed quantity a, then will x multiplied by any given number n or nx vaniſh in reſpect of a; or which is the ſame if x vaniſh in reſpect of a, ſo likewiſe will the quantity nx, bearing to x the aſſigned (or as it is called the finite) ratio of n to 1. For though at any particular point of time, x is a ſmaller fractional part of a than nx, yet x can be no aſſigned fractional part of a whatever, but by fur-

ther diminiſhing x the quantity nx may become the ſame fractional part of a that x was before. If then x may become equal to, or leſs than any aſſigned fractional part of a, ſo likewiſe may nx; that is, if x vaniſhes in reſpect of a ſo likewiſe does nx.

25. And for a like reaſon, if x vaniſh in reſpect of any quantity a, it will likewiſe vaniſh in reſpect of that quantity multiplied or divided by any number; or it will vaniſh in reſpect of a quantity bearing to a any aſſigned ratio, as that of m to 1. Thus if x vaniſh in reſpect of a, it will alſo vaniſh in reſpect of $3a$, $2a$, or $\frac{1}{2}a$, or $\frac{1}{3}a$. If any line vaniſh in reſpect to the diameter of a circle, it will likewiſe vaniſh in reſpect of the radius. For whatever part of the diameter x may be at any aſſigned point of time, let x further decreaſe till it be half what it was at that aſſigned time, and it will now be the ſame part of the radius that it was before of the diameter. By theſe rules we may frequently find when one quantity vaniſhes in reſpect of another propoſed quantity.

26. We have ſeen before that the ratio which two quantities bear to each other may have a limit although the terms them-

ſelves may increaſe or decreaſe perpetually without limit. If the terms approximate to each other, if the leſs never paſs the greater, and laſtly, if their difference vaniſh in reſpect of either term,* then the limit of their varying ratio is that of equality; and this whether the terms themſelves are ſuch as by increaſing do both of them become greater than any aſſigned quantity, or which is more common, ſuch as by decreaſing become leſs than any aſſigned quantity, or as it is uſually called, *infinitely little*. Becauſe the idea of the terms of a ratio is leſs abſtract, than that of the ratio itſelf, it is more uſual to ſay that the terms themſelves in this caſe are ultimately equal, though ſtrictly it is the ratio of the terms only that comes to a limit; for the terms themſelves are here ſuppoſed to increaſe or decreaſe without limit, or as we commonly ſay in infinitum.

27. That the difference between two quantities may thus vaniſh in reſpect of thoſe quantities; even though the quantities themſelves do vaniſh in reſpect of ſome aſſigned (or finite) quantity we ſhall ſhow by the following examples.

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* See par. 20, 21. 23.

LEMMA

28. The angle of reflection is equal to the angle of incidence.*

PROBLEM

29. Let a ray of light parallel to the axis EC (fig. 3) of a ſpherical concave ſpeculum fall on a given point A: To determine the interſection of the reflected ray with the axis of the ſpeculum

Let AC be the ſpeculum, E the center, EC the axis, QA the incident ray parallel to the axis EC and meeting the ſpeculum in A, let AT be the reflected ray interſecting the axis in T; draw the radius EA to the point of incidence and becauſe the angles AET and QAE †, alſo QAE and EAT ‡ are equal, the triangle EAT is iſoſceles. Draw DT perpendicular to EA and it will biſect the radius EA in D. Laſtly draw AP the ſine of the arch AC whoſe verſed ſine is PC. Then we have $EP : EA :: ED : ET$; call $EA = r$, $PC = v$, and $ED = \frac{1}{2}r$, therefore $r - v : r :: \frac{1}{2}r : \frac{\frac{1}{2}rr}{r-v} = ET$, and v vaniſhing $ET = \frac{1}{2}r$. Therefore biſect the radius EC in F, and if the incident ray QA be ſuppoſed always parallel to the axis EC and continually to approach to it, the interſection T of the reflected ray with the axis will continually approach to the point F as its limit.

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* Smith's Optics, Art. 8. † 29 El. 1. ‡ Lemma par. 28.

30. This is ſometimes proved thus, $EP : EA :: ED : ET$, but EP is to EA, ultimately in a ratio of equality * therefore ultimately $ET = ED$ or half the radius. This will in ſome ſort appear from the nature of the figure itſelf, for while the incident ray approaches to the axis, the baſe EA of the iſoſceles triangle EAT continues the ſame, but its height TD continually diminiſhes; therefore the ſides ET and TA will at laſt coincide with the baſe EA and becauſe they are equal, ET will at laſt be equal to half the baſe or half the radius.

31. But as this way of reaſoning may not ſeem very ſtrict, to ſhow the caſe clearly we will prove that the fixed quantity $EF = \frac{1}{2}r$ is the limit of the varying quantity $ET = \frac{\frac{1}{2}rr}{r-v}$.

The quantity ET always exceeds the limit EF, their difference FT being =

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* See par. 39.

$= \frac{\frac{1}{2} r r}{r - v} - \frac{1}{2} r = \frac{\frac{1}{2} r v}{r - v}$. Now *firſt* this difference continually decreaſes as (the arch AC and conſequently) v decreaſes: *ſecondly*; it never becomes negative: *thirdly*; it vaniſhes in reſpect of the varying quantity ET. For $ET : FT :: \frac{\frac{1}{2} r r}{r - v} : \frac{\frac{1}{2} r v}{r - v} :: r : v$, therefore as v vaniſhes in reſpect of r, ſo does FT in reſpect of ET the varying quantity; and therefore (as was before ſhown in par. 21) FT vaniſhes in reſpect of the fixed quantity EF.

32. We ſee then, that although the point T or interſection of the reflected ray with the axis, continually recedes from the ſpeculum while the incident ray by moving parallel to itſelf approaches to the axis, yet this point cannot recede in infinitum, not even though the incident ray be ſuppoſed to approach infinitely near the axis. The point F is the *limit* to which the point of interſection may *approach* nearer than by any aſſigned diſtance, *beyond* which it cannot *paſs*, and *at* which it can never *arrive* 'till (as we may ſay) the very interſection itſelf has loſt its being, by the coincidence both of the incident and reflected ray with the axis.

33. This point F is called the *focus* of parallel rays, and the interval FT the *aberration* of the reflected rays from their *geometrical focus*. Now although FT vaniſhes in reſpect of EF or half the radius, yet it does not vaniſh in reſpect of PC the verſed ſine. On the contrary it approximates to half the verſed ſine as its limit, or to ſpeak more ſtrictly, it is the ratio of FT to $\frac{1}{2}PC$ that approximates to the ratio of equality. For the aberration FT being equal to $\frac{\frac{1}{2}rv}{r-v}$, and $\frac{1}{2}PC$ being equal to $\frac{1}{2}v$, their difference is $\frac{\frac{1}{2}vv}{r-v}$ which never becomes negative; moreover $\frac{\frac{1}{2}rv}{r-v} : \frac{\frac{1}{2}vv}{r-v} :: r : v$. Therefore as v vaniſhes in reſpect of r, ſo does the difference between the aberration and half the verſed ſine in reſpect of the aberration and conſequently in reſpect of half the verſed ſine too. Nor is it ſtrange that the aberration which vaniſhes in reſpect of the fixed quantity $\frac{1}{2}r$, ſhould yet come to a limit with the varying quantity $\frac{1}{2}v$, ſeeing this laſt quantity does likewiſe vaniſh in reſpect of $\frac{1}{2}r$.

34. If we were to repreſent this in the

common conciſe way of ſpeaking, we ſhould ſay that the aberration $\frac{\frac{1}{2} r v}{r - v}$ when v vaniſhes becomes $\frac{\frac{1}{2} r v}{r}$ or $\frac{1}{2} v$, ſeeming to ſtrike v out of the denominator of the fraction as being nothing, and yet retaining it in the numerator as being ſomething. Indeed if v in the numerator be conſidered as nothing, the whole expreſſion vaniſhes, and the aberration is nothing, or rather there is no aberration; and thus the whole deſign of finding out this aberration falls to the ground.

It is from conciſeneſs of expreſſion in ſuch caſes as this, that mathematicians have been accuſed of unfair proceedings; of making the ſame quantity in the ſame expreſſion *to vaniſh or not to vaniſh* as ſuits the concluſion intended to be drawn from it: of conſidering quantities as ſomething in the beginning of a demonſtration, rejecting ſuch of them as they pleaſe at the end of it, and yet retaining the concluſion drawn from the original ſuppoſition.*

35. Another proof of the conciſe kind is as follows. $EA : EP :: ET : ED$, whence

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* See Analyſt, Section XIV and XV.

by diviſion of proportion $EA : (EA - EP)$ $PC :: ET : ET - ED = ET - EF = FT$; alternately $EA : ET :: PC : FT$, but EA is to ET ultimately as 2 is to 1,* therefore ultimately $2 : 1 :: PC : FT$ therefore $FT = \frac{1}{2} PC$.

36. But to proceed. The intire aberration is as we have ſeen $\frac{\frac{1}{2} r v}{r - v}$, from this ſubtract $\frac{1}{2} v$ and the difference $\frac{\frac{1}{2} v v}{r - v}$ may be called the *error of the geometrical aberration.* Now this quantity though it vaniſh in reſpect of the whole aberration yet comes to a limit with reſpect to the quantity $\frac{v v}{2 r}$ or a third proportional to the diameter and verſed ſine, or to uſe a more ſtrict way of ſpeaking, this error $\frac{\frac{1}{2} v v}{r - v}$ is to $\frac{v v}{2 r}$ ultimately in a ratio of equality. For the whole error of the aberration being $\frac{\frac{1}{2} v v}{r - v}$, and the third proportional aforeſaid being $\frac{v v}{2 r} = \frac{\frac{1}{2} v v}{r}$, their difference is $\frac{\frac{1}{2} v^3}{r \times \overline{r - v}}$, and the whole error is to this difference as

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* See par. 42.

r is to v, therefore this difference vaniſhes in reſpect of the whole error, and conſequently in reſpect of $\frac{\frac{1}{2}vv}{r}$. Thus then this third proportional to the diameter and verſed ſine may be called *the geometrical error of the geometrical aberration.* In like manner we may determine the error of this error and ſo on.*

37. To give an example of all this in numbers; let the radius of the ſpeculum be 30 inches, the arch AC eleven degrees and twenty nine minutes, whoſe verſed ſine is 2 when the radius is 100, conſequently when the radius is 30 inches this verſed ſine is 0,6 of an inch. We have therefore $r = 30$, $v = 0{,}6$, $\frac{1}{2}rr = 450$, $r - v = 29{,}4$, $\frac{\frac{1}{2}rr}{r-v} = \frac{450}{29{,}4}$ which (by common diviſion) is $15{,}3061224 = ET$ the whole diſtance of the point of interſection from the center, and this exceeds $EF = \frac{1}{2}r = 15$, or the geometrical focus by 0,3061224. Again, this quantity which is the whole aberration exceeds the geometrical aberration or $\frac{1}{2}v = 0{,}3$ by 0,0061224. Again, the diameter being 60, and the verſed ſine 0,6, we have 60 :

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* *Neque novit natura limitem.* Newton.

60 : 0,6 :: 0,6 : 0,006 = the third proportional to the diameter and verſed ſine; but 0,0061224 which is the whole error of the geometrical aberration, exceeds this third proportional by 0,000124 and ſo on.

38. All this will be confirmed by dividing the numerator of the original fraction $\left(\frac{\frac{1}{2}rr}{r-v}\right)$ by its denominator in the manner of compound diviſion in arithmetic, and ſo throwing the fraction into an infinite ſeries. For the quotient will be $\frac{1}{2}r + \frac{1}{2}v + \frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$, &c. where the law of the ſeries is manifeſt, each term being to the ſubſequent one always in the ratio of r to v.

Let then v vaniſh in reſpect of r and all the terms of the ſeries will vaniſh in reſpect of the firſt term; that is, the ſum of all the terms will in this caſe approximate to $\frac{1}{2}r$ as the limit; therefore the geometrical focus is $\frac{1}{2}r$.

Subtract now $\frac{1}{2}r$ from the whole ſeries expreſſing the diſtance ET, and we have the whole aberration $FT = \frac{1}{2}v + \frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$, &c. Let now v again be ſuppoſed to vaniſh in reſpect of r and all the terms in this ſeries vaniſhing in reſpect of the firſt we have $\frac{1}{2}v$ for the geometrical aberration.

Once more; ſubtract $\frac{1}{2}v$ from the ſeries expreſſing the whole aberration and we have $\frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$ for the whole error of the aberration. Let v again be ſuppoſed to vaniſh, and we have $\frac{\frac{1}{2}v^2}{r}$ for the limit of the error of the aberration, and ſo on.

Aſſuming $\frac{1}{2}r = 15$ and the ratio of r to v that of 100 to 2, as before

we have $\frac{1}{2}r = 15,$ *Geometr. Focus.*

$100 : 2 :: 15{,}0 : \text{to } \frac{1}{2}v = 0{,}3$ *Geometr. Aberr.*

$100 : 2 :: 0{,}3 : \text{to } \frac{\frac{1}{2}v^2}{r} = 0{,}006$ *Geometr. Error.*

$100 : 2 :: 0{,}006 : \text{to } \frac{\frac{1}{2}v^3}{r^2} = 0{,}00012$

$100 : 2 :: 0{,}00012 : \text{to } \frac{\frac{1}{2}v^4}{r^3} = 0{,}0000024$

Sum of the terms $= ET = 15{,}3061224$ which is true to ſeven places of decimals as was before found by diviſion.

PRO-

PROPOSITION

39. In a circle whoſe center is *C* (fig. 4) radius *CA*, diameter *Aa*, let *AB* be the chord, *FB* the ſine, *AD* the tangent of the arch *AB*. I ſay, that while the arch *AB* continually decreaſes without limit, *firſt* the ſine continually approximates to the tangent. *Secondly* the ſine never exceeds the tangent; *thirdly* their difference will vaniſh in reſpect of either ſine or tangent.

Firſt. *BF* : *DA* :: *CF* : *CA*, but while the arch *AB* decreaſes *CF* approximates to *CA*, therefore *BF* approximates to *DA*. *Secondly*. *CF* can never exceed *CA*, therefore *BF* can never exceed *DA*. *Thirdly*. The arch continually decreaſing without limit vaniſhes (in our ſenſe of the word par. 20) in reſpect of the diameter which is fixed; therefore the chord *AB* which is leſs than the arch, likewiſe vaniſhes in reſpect of the diameter *Aa*. But *Aa* : *AB* :: *AB* : *AF*, therefore if *AB* vaniſhes in reſpect of *Aa*, *AF* will vaniſh in reſpect of *AB*, and much more will *AF* vaniſh in reſpect of *Aa*: but if *AF* vaniſh in reſpect of *Aa*, it will alſo vaniſh in reſpect of $\frac{1}{2}Aa$ or *CA* (by par. 25). Now *CA* : *CF* :: *AD* :

FB, and by diviſion of proportion, $CA : AF :: AD : AD - FB$. As therefore AF vaniſhes in reſpect of CA, ſo does $AD - FB$ (the difference of the ſine and tangent) vaniſh in reſpect of AD the tangent, and conſequently in reſpect of FB the ſine, by par. 23.

We ſee then that while the arch continually decreaſes without limit, the ſine approximates to the tangent; *ſecondly*, the ſine never exceeds the tangent; *thirdly*, although the ſine and tangent both vaniſh in reſpect of the radius, yet their difference vaniſhes in reſpect of theſe quantities themſelves. Therefore the ratio of equality is the limit of the varying ratio which the ſine and tangent have to each other, while they both decreaſe perpetually without limit; or it is their *ultimate ratio*; or we may ſay that they are *ultimately equal.*

40. The ſine is leſs than the chord; for in the right angled triangle AFB, the ſide BF is leſs than the hypothenuſe AB. The chord is leſs than the arch; this is ſelf-evident, the chord being a ſtraight line, and the arch a curve, both terminated by the ſame points A and B. The arch is leſs than the tangent. For from the point D draw

another tangent to the circle in *H*; and the lines *ABH*, *ADH* will be terminated by the ſame points *A* and *H*, and will have their concavities turned the ſame way: therefore the included arch *ABH* will be leſs than the ſum of the two equal tangents *DA* and *DH*; * conſequently half that arch or *AB* will be leſs than half the ſum of the tangents or *AD*.

41. *Cor*. 1. Hence the ſine chord arch and tangent are all ultimately in a ratio of equality. This may appear becauſe the chord and arch are included between the ſine and tangent, but perhaps more plainly thus. Of theſe four quantities, *viz*. the ſine chord arch and tangent, the ſine is the leaſt, and the tangent the greateſt, by the laſt paragraph; therefore the difference between the ſine and tangent is greater than the difference between any other two of thoſe four quantities. If therefore the greateſt of all thoſe differences, *viz*. that between the ſine and tangent vaniſhes in reſpect of the leaſt of all thoſe quantities namely the ſine; much more will the difference between any other two of thoſe four quantities vaniſh in reſpect of the quantities themſelves.

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* *Archimedes De ſphæra & cylindro. Defin.* 2 *& axiom.* 2 or *Mac Laurin*'s Fluxions, art. 180 and 183.

42. *Cor.* 2. Join *B a* and in the right angled triangle *B F a*, the hypothenuſe *B a* is greater than the ſide *F a*; therefore *A a* — *B a* is leſs than *A a* — *F a* or *A F*; but by par. 39 while the arch *A B* decreaſes perpetually without limit, *A F* vaniſhes in reſpect of *A a*; much more then does *A a* — *B a*, the difference between *A a* and *B a* vaniſh in reſpect of *A a*, therefore the ultimate ratio of *A a* to *B a* is that of equality; and the ultimate ratio of $\frac{1}{2}$*A a* or *B C* to *B a*, is that of 1 to 2.

SCHOLIUM

43. The reader will find a different proof of this celebrated propoſition in *Saunderſon*'s Fluxions, page 248, and another in *Morgan*'s notes on *Rohault*, publiſhed by Dr. *S. Clarke*, Part II. Chap. 28, and re-publiſhed at *Cambridge* 1770, ſee page 28. The beſt proof of all others is undoubtedly Newton's own proof. *Princip. Lib.* I. *Lemma* VII. That demonſtration being very conciſe, I ſhall give it here, and enlarge upon the ſeveral ſteps.

Let *A C B* (fig. 13) be an arch of a circle whoſe center is *R*, chord *A B*, tangent *A D*, ſecant *R D*. Let the tangent *A D* be produced to a diſtant point d, and through d draw dr parallel to the ſecant DR, cutting the line AR (produced indefinitely) in r; then if with the center r and radius Ar an arch Acb be deſcribed meeting dr in b, *firſt* it will be ſimilar to the arch ACB, (though not ſimilarly ſituated) * that is, the arches Acb and ACB will have each the ſame proportion to the circumferences of their reſpective circles, becauſe the angles Ard and ARD are equal by conſtruction.† *Secondly*, the chord AB produced, will paſs through b, and ABb will be the chord of of the arch Acb: for the arches Acb and ACB being ſimilar, their chords AB and Ab will make equal angles with their common tangent ADd and therefore coincide. *Laſtly*, the lines AB, AD and arch ACB are always proportional to the lines Ab, Ad and arch Acb.

$$\left.\begin{array}{r}\text{For the line } AB\\ \text{the line } AD\\ \text{the arch } ACB\end{array}\right\}\text{is to the}\left\{\begin{array}{l}\text{line } Ab\\ \text{line } Ad\\ \text{arch } Acb\end{array}\right\}\begin{array}{l}\text{as the radius } AR,\text{ is to}\\ \text{the radius } Ar,\text{ or as } AD\\ \text{is to } Ad.\end{array}$$

therefore alternately $AB : AD :: Ab : Ad$, and likewiſe $AB : ACB :: Ab : Acb$: that is, the lines AB, AD and arch ACB are to each other, as the lines Ab, Ad and arch Acb are to each other. This premiſed, imagine now the ſecant RBD to turn on the center R, while the point B continually approaches to the point A. At the ſame time imagine the line dr to turn round the fixed point d in the contrary direction, but with the ſame angular motion ſo as always to keep parallel to RD; by this means the center r will recede from A more and more in infinitum, and the arch Acb will be a portion of a circle continually growing larger and larger; or as we may ſay, the arch Acb will continually unbend itſelf. In the mean time the chord ABb turning round the point A, continually increaſes, and as the line dr approaches to paralleliſm with the line Ar, the increaſing chord Ab approximates to the invariable tangent Ad and carries the intermediate arch Acb along with it. Now the point B approaching to the point A the angle BAD continually decreaſes, and there is no limit to its decreaſe by *prop.* 16. *El.* 3, therefore neither is there any limit to the decreaſe of the angle bAd nor to the approach of Ab to Ad, therefore the lines Ab, Ad and the intermediate arch Acb will ultimately coincide and be equal; whence the chord AB, tangent AD and arch ACB, which are to each other as Ab, Ad

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* *Mac Laurin*'s Fluxions, art. 122.

† 29 El. 1.

Ad and *Acb*, will be ultimately to each other in a ratio of equality.

We may obſerve, that the lines *Ab*, *Ad* are always finite. *Ad* is of a fixed invariable length; *Ab* approximates to *Ad*, can never exceed *Ad*, nor indeed become equal to it 'till *db* is parallel to *AR*; but the finite lines *Ab* and *Ad* can approach nearer to equality than by any aſſigned difference; that is, the difference between *Ab* and *Ad* vaniſhes in reſpect of thoſe (finite) lines; whence this is inferred of the lines *AB*, *AD* always proportional to them.

Newton's artifice conſiſts in ſhowing us a finite part of an infinitely large circle which we *can* ſee; and arguing from it analogically to an infinitely ſmall part of a finite circle which we *can not* ſee.

The foregoing explanation reſtraining the Lemma to circles, puts a ſenſe upon it more confined than Newton intended, but it will be the eaſier underſtood by beginners.

To propoſe the Lemma in more general terms, we muſt firſt premiſe, that the curve *ACB* is ſuppoſed to have continual curvature at *A*, and though not a circle, yet it is ſuppoſed to have circular curvature at *A*,

A,[*] that ſo the angle between the chord and tangent (while the point B approaches to A) may become leſs than any (aſſigned) right lined angle.

In the demonſtration the point d is ſuppoſed to be fixed, the line Ad being of a given length. *Secondly*, The line bd is always to be parallel to the ſecant BD. *Thirdly*, The two curves ACB and Acb are to have one common tangent at A. *Laſtly*, The arch Acb is to be always ſimilar to the arch ACB, that is, ACB and Acb are to be ſimilar parts of ſimilar figures, the linear magnitude of Acb being to that of ACB, as Ad is to AD; moreover, A repreſents correſponding points of the two figures.[†] For inſtance; if ACB be part of a circle, then muſt Acb be alſo part of a circle: The radius of Acb muſt be to the radius of ACB as Ad is to AD, and then the part Acb, cut off by the line db, and the part ACB will be ſimilar arches of two circles. If ACB be a Parabola then muſt Acb alſo be a Parabola, whoſe parameter is to the parameter of ACB, as Ad is to AD. If ACB be an

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[*] *Cæterum in his omnibus ſupponimus curvaturam ad punctum* A, *nec infinitè parvam eſſe, nec infinitè magnam. Vide Scholium ad Lemma* XI. *Lib.* I. *Princip.*

[†] *Mac Laurin*'s Fluxions, art. 122.

Ellipſe then muſt *Acb* be an Ellipſe. To make make theſe Ellipſes ſimilar figures, the proportion of the greater axis to the leſs, in each Ellipſe, muſt be the ſame; and that the arch *A c b*, cut off by the line *d b*, may be ſimilar to the arch *ACB*, the greater axis of the latter Ellipſe muſt be to the greater axis of the former, as *Ad* is to *AD*; moreover, if the point *A* is the extremity of the greater axis in one Ellipſe, it muſt likewiſe be ſo in the other.

It follows from the ſimilarity of the arches *ACB* and *Acb*, *Firſt*, that their ſubtenſes *AB* and *Ab* make equal angles with their common tangent *ADd*, and therefore coincide. *Secondly*, that the lines *AB*, *AD*, and arch *ACB* are always to each other as the lines *Ab*, *Ad*, and arch *Acb* are to each other. This premiſed, while the point *B* approaches to *A*, the chord *Ab*, turning round the point *A*, approximates to the invariable tangent *Ad*, and carries the intermediate arch *Acb* along with it. As the angle *DAB* decreaſes without limit, or in other words ultimately vaniſhes, ſo the angle *dAb* ultimately vaniſhes, and the finite lines *Ab*, *Ad*, and arch *Acb*, ultimately coinciding, are equal; therefore the evaneſcent lines *AB*, *AD* and arch *ACB*, (which are always to each other as *Ab*, *Ad* and *Acb*) will have for their ultimate ratio, that of equality.

The angle *ADB* is always ſuppoſed to be finite, but it may be either conſtant or variable. The angles *ADB* and *Adb* may if you pleaſe be ſuppoſed always right angles, or any other given angle; the demonſtration will ſtill be the ſame.

LEMMA I.

44. If a body be acted upon by a force which is every where as the diſtance from the center *A*, (fig. 5) the time of its deſcent to that center will be the ſame from whatever place it falls. *Princip. Lib.* I. *Prop.* XXXVIII. *Cor.* 2.

LEMMA II.

45. If a body be acted upon by a force which is every where as the diſtance from the center *A*, (fig. 5) the time of deſcent through any ſpace *PA*, is to the time in which it would deſcend through that ſame ſpace if impelled by half the firſt force (at *P)* uniformly continued, as the circumference of a circle is to four diameters.

For the time of deſcent through the ſpace *PA* is equal to the time in which the body revolving by the force at *P*, in a circle whoſe radius is *PA*, would deſcribe a quadrant, *Princip. Lib.* I. *Prop.* XXXVIII. *Cor.* 1. and in this time by the ſame force uniformly continued, the body would deſcend through a ſpace which is a third proportional to the diameter of the circle and quadrant thus deſcribed, *Princip. Lib.* I. *Prop.* IV. *Cor.* 9; that is calling the diameter D and quadrant $\mathcal{Q}$, a ſpace equal to $\frac{\mathcal{Q}\mathcal{Q}}{D}$; but the time of deſcent through this ſpace $\frac{\mathcal{Q}\mathcal{Q}}{D}$ by the uniform force at *P*, is to the time of deſcent through the ſpace $PA = \frac{1}{2}D$ by half that uniform force, in a ſubduplicate ratio of thoſe ſpaces directly, and a ſubduplicate ratio of thoſe forces inverſely;* that is directly as $\sqrt{\frac{\mathcal{Q}\mathcal{Q}}{D}}$ is to $\sqrt{\frac{1}{2}D}$, and inverſely as $\sqrt{1}$ is to $\sqrt{\frac{1}{2}}$, or compounding theſe ratios, as $\mathcal{Q}$ is to D, or as $4\mathcal{Q}$ is to $4D$, or as the circumference of a circle is to four diameters.

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* Introduction to *Saunderſon*'s fluxions. Prop. VII. Cor. IV. Can. 3.

PROPOSITION I.

46. The accelerating ſorce of a body oſcillating in a circle, is to its abſolute weight, as the ſine of its angular diſtance from the loweſt point, is to the radius.

Let *P* (fig. 6) be the body oſcillating in the arch of a circle *PAp*, whoſe loweſt point is *A*, and center *C*. Let *PF* be the ſine of the arch *PA*, and let *PT* be a tangent to the circle at *P* meeting the radius *CA* produced in *T*, then will the accelerating force of the body at *P* be the ſame as if it were placed on the tangent *PT* conſidered as an inclined plane, and will therefore be to its abſolute weight, as the height of that plane is to its length, * or as *FT* is to *PT*, or becauſe of the ſimilar triangles *PFT* and *CFP*, as *PF* is to *CP*, or as the ſine of the arch *PA* (the meaſure of the angular diſtance *PCA*) is to radius.

47. *Cor.* 1, In the ſame circle the accelerating force is every where as the ſine of the arch *PA*, but becauſe the ſine and arch approximate to the ratio of equality while the arch decreaſes, therefore the ratio of

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* Introduction to *Saunderſon*'s fluxions, Prop. VII, or *Keil*'s Phyſics, Lect. XV. Theorem 35.

the ſines to each other will approximate to, and be ultimately equal to the ratio of their correſponding arches to each other. Wherefore ultimately, the accelerating force is as the arch *PA*, the diſtance of the body from the loweſt point *A* meaſured along the circle in which it oſcillates.

48. *Cor.* 2. Hence the times of deſcent down unequal arches *PA*, *eA*, approximate to equality while thoſe arches decreaſe in infinitum, and (as we may ſay) ultimately thoſe times will be equal. For when the accelerating force is as the diſtances *PA*, *eA*, the times of deſcent will be equal from whatever place the body begins to deſcend, by par. 44.

PROPOSITION II.

49. All things as before, draw *PA* the chord of the arch *PeA* (fig. 7); I ſay that the accelerating force of the body upon the arch at *P*, is to the accelerating force of the body placed any where* on the chord *PA*, ultimately as 2 to 1.

Draw the diameter *aCA*, join *aP* and *CP*, let *AT* be a tangent to the circle at *A* and PT a tangent to the circle at P, let theſe tangents meet in T; then conſidering the chord PA and tangent PT as inclined planes whoſe heights are equal, the accelerating force on the plane PT (equal to that on the arch at P) will be to the accelerating force on the plane PA, as PA is to PT,[*] or becauſe of the ſimilar triangles PAT and PaC,[†] as Pa is to PC, that is ultimately as 2 is to 1.[‡]

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* Introduction to *Saunderſon*'s fluxions, Prop. VII. Cor. I. or *Keil*'s Phyſics, Lect. XV. Theorem 35.

50. *Cor.* 1. The time of deſcent through the arch PeA is to the time of deſcent through the chord PA, ultimately, as the circumference is to four diameters. For ultimately the length of the arch and chord are equal, and ultimately the accelerating force of the body on the arch is every where as its diſtance from the loweſt point A, by par. 47; but the accelerating force of the body on the chord is uniform and equal to half the firſt force on the arch at P by this propoſition, therefore by par. 45, the time of deſcent through the arch, is to the time of deſcent through the chord, as the circumference is to four diameters.

51. *Cor.* 2. The time of one oſcillation

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[*] Introduction to *Saunderſon*'s fluxions, Prop. VII. Cor. I. or *Keil*'s Phyſics, Lect. XV. Theorem 35.

[†] 32 El. 3.

[‡] Par. 42.

is to the time of deſcent down half the length of the pendulum *ultimately*, as the circumference of a circle to its diameter. For the time of deſcent through the chord *PA* being always equal to the time of deſcent down the diameter *aA*,* we have *ultimately*, the time of deſcent through the arch, to the time of deſcent down the diameter, as the circumference is to four diameters by the laſt corollary; whence the deſcent and aſcent along the arch, or the time of one oſcillation, is to the time of deſcent down the diameter, as the circumference to two diameters; but the deſcent down the diameter, is to the deſcent down $\frac{1}{4}$ of the diameter, or half the length of the pendulum, as 2 to 1; therefore (compounding theſe ratios) the time of one oſcillation, is to the time of deſcent down half the length of the pendulum, as the circumference of a circle to its diameter.

SCHOLIUM

52. If the ſines increaſed in the ſame ratio with the arches or diſtances *PeA*, all oſcillations would be preciſely equal; but as the ſines increaſe in a leſs ratio than their

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* *Keil's* Phyſics, Lect, XV. Theorem 39.

arches, the accelerating force when the arch is enlarged, is not ſufficiently increaſed to make the times of deſcent and conſequently the oſcillations equal; therefore the oſcillations through larger arches take up a longer time. Contrariwiſe, if the arch be leſſened, the time of deſcent is leſſened, and therefore the time of deſcent and aſcent, or the time of one oſcillation is leſſened. But though the arch may decreaſe in infinitum without any limit, yet the time of an oſcillation through that arch will not decreaſe proportionably nor in infinitum, but has a limit. Thus if the length of the pendulum be 39,2 inches, the time of one oſcillation through any arch will be ſomewhat more than one ſecond. Now though that arch may be taken leſs than any aſſigned arch, yet the time of one oſcillation will not become leſs than any aſſigned time. For if the pendulum oſcillates at all, it muſt take up a dederminate finite time, namely one ſecond, how ſmall ſoever the arch may be, through which it oſcillates. Thus then although ſtrictly ſpeaking, there is no ſuch thing as the time of a *leaſt oſcillation*, (there being no *leaſt arch* and therefore no leaſt oſcillation;) yet there is ſtrictly ſpeaking a *leaſt time* of the oſcillation of every pendulum; and the proportion of this *leaſt time* to the time of deſcent down half the length of that pendulum has been before pointed out.

53. The time here ſought is that which is the limit of the varying time of an oſcillation through an arch ſuppoſed to decreaſe continually without limit; therefore in comparing thoſe quantities on which the time of one oſcillation depends, it is the limit of their varying ratio, or rather the limit of the varying ratio of the length of thoſe lines to which ſuch quantities are analogous, that muſt be attended to. *

In the firſt corollary to the firſt propoſition par. 47, when we had found the accelerating force to be always as the ſines, we thence concluded it to be ultimately as the arches, becauſe of the ultimate equality of the ſine and arch. In the firſt corollary to the ſecond propoſition par. 50, we compared the time of deſcent along the arch, with the time of deſcent along the chord, and as far as thoſe times depend on the length of the line deſcribed, (and not on the accelerating force) we conſidered them as ultimately equal, becauſe of the ultimate equality of the length of the arch and chord. In both theſe caſes then we admitted this propoſition concerning the ultimate equality of the ſine chord and arch; and it was rightly applied becauſe the ratio of the quantities to be compared, (the forces in the former caſe, par. 47, and the times in the latter, par. 50) depended on the ratio of the *length* of theſe lines; but we miſunderſtand the propoſition, if we ſuppoſe it to imply that the arch and chord do ultimately *ſo coincide* that they may be conſidered as *one and the ſame line*. For on whatever grounds we ſuppoſe the arch thus to coincide with the chord, we might likewiſe ſuppoſe it to coincide with the ſine *PF*, or with the tangent to the loweſt point *TA* (fig. 8) for theſe are all *ultimately equal*; but both ſine and tangent in this caſe are parallel to the horizon and therefore there would be no accelerating force, no oſcillation at all; and thus the whole enquiry would fall to the ground. On the contrary; the arch is a curve all whoſe parts have different inclinations to the horizon, but the chord is a right line which has every where the ſame inclination. The accelerating force on the arch continually increaſes with the diſtance from the loweſt point, but the force on the chord is every where the ſame. Neither the law of the accelerating force, nor the quantity of it at the beginning of the deſcent in theſe two caſes approximate by the decreaſe of the arch; we have therefore no reaſon to expect that the times of deſcent ſhould approximate, notwithſtanding the lengths of the lines deſcended through, do approximate and become ultimately equal. *Keil* and others after him ſay that the *arch and chord differ but little either in length or declivity, therefore neither do the times of deſcent.** It ſhould be ſhown that the times are in ſome finite ratio of theſe quantities, otherwiſe ſmall differences in theſe circumſtances may occaſion great differences in the times of deſcent. The aſſertion (rightly explained) is true with regard to the length of the chord and arch, but not with reſpect to their declivity, if we allow that any compariſon can be made between the declivity of a curve and a right line, which yet ſeems to be abſurd.

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* It may be obſerved, that in this and ſimilar caſes, the very nature of the problem ſhows that it is the limit of the varying quantity, or the limit of the varying ratio that we are to ſeek; as will be further exemplified in par. 55.

54. The ratio of the length of the arch

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* *Keil's* Phyſics, Lect. 15. Theorem 41. *Monſ. Parent*, I believe was the firſt that fell into this error.

to the ſine, or of the arch to the chord which varies while the arch decreaſes, *has for its limit the ratio of equality*, the difference of their lengths vaniſhing in reſpect of the lengths themſelves, all which is only expreſſed in other words, when we ſay that they are *ultimately equal*; but the arch is ever to be conſidered as a CURVE, and the chord as a RIGHT LINE, two ideas which ought not to be confounded.*

55. We ſaid before that in problems of this

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* Newton ſays — *figura rectilinea, quæ chordis evaneſcentium arcuum comprehenditur, coincidit ultimo cum figura curvilinea. Princip. Lib.* I. *Lemma* III. *Cor.* 2. —— This is the expreſs language of indiviſibles, and muſt not be underſtood literally any more than *ſumma ultima* in the preceding corollary. Newton himſelf expounds *ſumma ultima* and *ratio ultima* by *limites ſummarum & rationum*, and reſts the whole force of his demonſtrations upon the doctrine of LIMITS. *Scholium ad Lem.* XI. *pag.* 37. *Ed.* 3.

Newton frequently expreſſes himſelf in ſuch terms as were uſed by the writers on indiviſibles in his time; and though at the end of that remarkable ſcholium he cautions his reader againſt being miſled by the uſe of thoſe terms, yet he ſeems ſometimes to forget that caution himſelf, of which theſe corollaries are an inſtance. The very expreſſion — *ultimæ quantitatum evaneſcentium rationes*, (or ultimate ratios) taken literally implies the ratios of quantities in an ultimate ſtate, that is, of quantities infinitely ſmall or infinitely great: and though Newton afterwards ſays that by ultimate ratios is meant the limit of the varying ratio of decreaſing quantities, and not the ratio of ultimate quantities, yet he ſtill ſpeaks of varying ratios as reaching their limit — *quum quantitates diminuuntur in infinitum*, — when the terms become infinitely little.

It will be ſaid that all this is only to cavil about words; — ſo it is, with thoſe who interpret rightly this mode of ſpeaking; but it is what will certainly miſlead common readers, and almoſt as certainly prove a ſnare to the very ableſt writer, if he is not continually upon his guard.

ſort, their nature ſhows that it is the limit of ſome varying ratio that we are to ſeek, not the ratio itſelf. Thus if the areas of two curvilinear ſpaces are to be compared together, let their baſes be each divided into the ſame number of parts and on thoſe diviſions let parallelograms be drawn ſo as to form right lined figures inſcribed within the curvilinear ones, as is directed in the fourth lemma of the firſt book of the *Principia.* Now both the ratio of theſe parallelograms (each to each) and of the right lined figures compoſed of them all will vary, as the number of parts varies into which the baſe of each figure is divided, and the limit of the varying ratio (when it can be found) is that of the curvilinear figures themſelves, as is proved in the lemma aforeſaid, and a very neat example of finding that limit in a particular caſe is given by *Saunderſon* in his fluxions, page 243. The nature of the problem (*viz.* to find the ratio of the curvilinear figures) ſhows plainly that it is not the ratio of the inſcribed rectilinear figures that we are to ſeek, how great ſoever the number of the little parallelograms that compoſe them may be, but the limit of that (varying) ratio.

56. It may be worth while to illuſtrate this by another inſtance in the ſolution of the following

PROBLEM

To find the proportion of the velocities of two flowing lines at a given inſtant, from their known cotemporary increments.

Let us for the preſent ſuppoſe the velocities of theſe lines to be continually accelerated; now if the time in which their cotemporary increments are generated be ſuppoſed perpetually to decreaſe, the ratio of thoſe increments will continually vary; and the limit of their varying ratio is that of the velocities ſought.

For every ſuch increment may be diſtinguiſhed into two parts; *firſt*, that which generated by the velocity of the flowing line at the given inſtant, continued uniformly for the whole time in which the increment is acquired; and *ſecondly*, that part of it which is generated in conſequence of the ſubſequent increaſe of the firſt velocity. Now this additional velocity grows gradually from nothing, therefore by leſſening the time in which the increment is produced, this additional velocity will continually leſſen, and will become leſs than any aſſigned fractional part of the firſt velocity which the line had at the given inſtant. Hence the latter part of the increment, generated in conſequence of the additional velocity, will ultimately vaniſh in reſpect of the former part of the increment generated by the firſt velocity immutably continued, even though by thus diminiſhing the time the former part of the increment does itſelf vaniſh in reſpect of any aſſigned or finite line. In any ſuch flowing line then, if the whole increment be compared with that part of it which is generated by the firſt velocity uniformly continued, *firſt* theſe two quantities continually approximate to each other as the time in which they are generated decreaſes; *ſecondly*, the one can never paſs beyond the other, for the whole increment can never become leſs than a part of it: *laſtly*, their difference, namely the part generated in conſequence of the acceleration, vaniſhes in reſpect of the quantities themſelves; therefore by par. 26, the limit of their varying ratio is that of equality.

In each line then the whole increment is to the former part of it (generated by the velocity at the given inſtant uniformly continued) ultimately in a ratio of equality, and alternately the whole increment of one flowing line is to the whole cotemporary increment of the other flowing line, ultimately, as the former part of the increment of the firſt flowing line is to the former part of the increment of the other flowing line, that is, as the velocity of the firſt flowing line at the given inſtant, is to the velocity of the other flowing line.

57. Here the nature of the problem directs us to ſeek the *limit* of the varying ratio of the cotemporary increments of theſe lines, (varying as the time in which they are generated decreaſes perpetually without limit) not to ſeek the ratio of the increments themſelves, how ſmall ſoever we may ſuppoſe the time to be in which they are generated.

58. Having ſhown this in general, we may now proceed to enquire what the limit of that varying ratio is in any particular given inſtance. For example, let the former of theſe two lines be called x, and let the latter line be a third proportional to a fixed line (aſſumed at pleaſure) called unity and to the former line x, then will its value

be xx; that is, x and xx will repreſent any two cotemporary values of lines having this particular relation to each other. Let v be an increment of the line x, then will $x+v$ and $\overline{x+v}|^2 = xx + 2xv + vv$ be two other cotemporary values of theſe flowing lines, whence their cotemporary increments will be v and $2xv + vv$ whoſe ratio is that of 1 to $2x+v$. Now ſuppoſing the time in which the increment v is generated to decreaſe continually without limit, the value of v will likewiſe decreaſe without limit, and will ultimately vaniſh in reſpect of the finite line $2x$, which is not affected by leſſening the increment v, ſo that $2x+v$ will be to $2x$ ultimately in a ratio of equality. Hence the limit of the varying ratio of 1 to $2x+v$ while v decreaſes without limit, will be that of 1 to $2x$, which is therefore the proportion of the velocities ſought. Therefore it will be, as the fixed line called unity is to twice the flowing line x, ſo is the velocity of the line x to the cotemporary velocity of the line xx.

59. The whole argument will be the ſame *(mutatis mutandis)* if we ſuppoſe the velocities of theſe lines to be continually retarded. The increment may in that caſe

be diſtinguiſhed into two parts, that which would be generated by the firſt velocity uniformly continued and that which is loſt by the retardation. Now when the time is continually diminiſhed, this latter part will ultimately vaniſh in reſpect of the former part as before. If we ſuppoſe the velocities of the flowing lines to be uniform, then their cotemporary increments are the meaſures of thoſe velocities.

60. *Cor.* If the velocity of the line x be repreſented by $\dot{x}$, the cotemporary velocity of the line xx will be repreſented by $2x\dot{x}$. For 1 is to $2x$ as $\dot{x}$ is to $2x\dot{x}$.

ESSAY II.

On the POWER *of the* WEDGE.

1. LET *PEH* (fig. 9) be a rectangular wedge, whoſe angular point is *E*, face *PE*, baſe *EH*, and back *PH*. Let this wedge ſlide freely on the plane *LN*. Let a body *E* be urged in the direction *KE* againſt the face of the wedge, and let that body be kept in the direction *KE* either by bearing againſt a plane parallel to *KE*, or by a ſtring *EB* always drawing the body at right angles to *KE*; or let the body be in any manner prevented from moving along *PE* and compelled to move in the line *KE*, by a force acting at right angles to *KE*.

2. Given the angle of the wedge *PEH*, and the angle *KEP* which the direction of the force upon *E* makes with the face of the wedge: To find the proportion between

the force applied to the body in the direction *KE*, and the force applied perpendicularly to the back of the wedge, when theſe two forces balance each other, ſo that the wedge moves neither forward nor backward. To find likewiſe the proportion of the forces aforeſaid, to the preſſure of the wedge againſt the plane *LN* and to the preſſure of the body *E* againſt the plane parallel to *KE*, occaſioned by the action of the other two forces when they balance each other.

Before we enter upon the ſolution of the problem, it may be proper to premiſe the following mechanical principles univerſally received.

3. *Firſt*, When a body is acted upon by one force only, it cannot be ſuſtained by another ſingle force, unleſs the latter be both equal and *oppoſite* to the former.

4. *Secondly*, When two or more forces act at once upon a body, the combined effect of them all is to move it in one direction only; and they will have the ſame effect and be equivalent to a ſingle force acting in one particular direction. *

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* The rule for finding the quantity of that force and the particular direction in which it muſt act, is in almoſt every book of mechanics. See the introduction to *Saunderſon's* fluxions.

5. *Thirdly*, When an hard body acts *in any direction whatever* on the ſurface of another, and ſuch ſurface is perfectly hard and ſmooth; whether the former body preſſes continually on the latter, or only ſtrikes it, yet the effect of that preſſure in one caſe, or of the impact in the other, and the conſequent motion in the body *acted upon*, (ſuppoſing it at liberty,) is always in a DIRECTION PERPENDICULAR TO THAT SURFACE. *

6. Theſe principles laid down it is manifeſt that when the body *E* is both urged in the direction *KE* and at the ſame time ſuſtained in the line *KE* by a plane parallel to *KE*, the original force in the direction *KE* combined with the reaction of the ſuſtaining plane (or with the force of the ſtring *EB* drawing at right angles to *KE*) may be conſidered as a ſingle force urging the body *E* againſt the face of the wedge: but the

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* When a plane perfectly hard and ſmooth is ſuſtained againſt the preſſure of an hard body, it is ſaid to *react* againſt the body that preſſes it, and as the ſuſtaining force muſt be in a direction oppoſite to that in which the plane would move if at liberty, the reaction of the plane is therefore perpendicular to its ſurface.

The force that ſuſtains ſuch a plane muſt be equal to the force of the body that preſſes it, that is, the reaction of the plane againſt the body, is equal to the action of the body upon the plane.

The principle here laid down is further explained in par. 21.

action of an hard body *E*, againſt the ſmooth and hard ſurface of the wedge is always perpendicular to its face *PE*, and a ſingle force impreſſed on the wedge in the oppoſite direction, *and in that only*, will counteract the other two (combined) forces and keep the wedge in equilibrio. Now the directions of theſe three forces being known, their proportions when they balance each other are eaſily found by a common rule in mechanics. For through *P* draw *PA* parallel to *KE*, through *E* draw *ED* perpendicular to *KE* meeting *PA* in *D*, alſo draw *EA* perpendicular to *PE* meeting *PA* in *A*; and *ED*, *DA*, *AE* the ſides of the triangle *EDA*, lying in the direction of the forces aforeſaid are to each other as thoſe forces. *

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* It may perhaps throw ſome light upon this ſubject if we conſider the whole effect of the original force in the direction *KE*, and the whole effect of the reaction of the ſuſtaining plane, ſeparately from each other.

Now when a force acts upon the body *E* in a direction *KE* oblique to the hard and ſmooth ſurface *PE*, ſuch a force will produce in the body *E* a tendency to move along that ſurface *PE*, (down the face of the wedge) and likewiſe a preſſure of the body *E* againſt the wedge, perpendicular to its face *PE*. The body *E* is here ſuppoſed to be kept from moving down the face of the wedge by another force, namely the reaction of the ſuſtaining plane: This force is perpendicular to the former force *KE*, therefore likewiſe oblique to the ſurface *PE*. This force then, being in the oblique direction *ED*, partly urges the body *E* up the face of the wedge, and partly adds to the perpendicular preſſure of the body *E*, againſt the wedge.

7. *EA* then repreſents the quantity and direction of the force impreſſed on the wedge by the body *E*, but the effort of the wedge to move in the direction *EA* is not in our caſe ſuſtained by a ſingle force equal and oppoſite, but by two others, one the reaction of the plane on which the wedge ſlides and the other the force on the back. The force *EA* being partly oppoſed to the plane *LN*, and partly to the back *PH*, will produce partly a preſſure of the wedge upon the plane *LN* which is balanced by the reaction of that plane, and partly a tendency in the wedge to move backwards, which is balanced by the force applied to the back. Now the directions of theſe two forces as well as that of *EA* are known, whence their proportions when they balance each other will be thus found. From *A* draw

wedge. From *D* let fall *Dd* perpendicular to *EA*; and if *DA* as before repreſents the original force applied to the body *E* in the direction *KE*, then will *Dd* be the force urging the body down *PE*, and *dA* the force urging the body againſt the wedge. Again, if the ſuſtaining plane prevents the body from moving down the face of the wedge, that part of the reaction of this plane which is in the direction *EP* muſt be equal and contrary to *Dd*, and therefore is repreſented by *dD*; whence *ED* will repreſent the whole reacting force of that plane, and *Ed* will repreſent the other part of this force which urges the body perpendicularly againſt the face of the wedge. To *Ed* add *dA*, and their ſum *EA*, will be the whole force with which the body *E* preſſes the face of the wedge, as before.

For the nature and laws of oblique forces, ſee par. 21.

draw *AF* perpendicular to *EH*; and *EA* repreſenting as before the quantity and direction of the force impreſſed on the wedge; *FA* will repreſent the preſſure on the plane *LN*, and *FE* the force to be applied perpendicularly to the back of the wedge to keep it in equilibrio.

8. If then *DA* repreſents the force applied to the body *E* in the direction *KE*; *FE* will be the force applied perpendicularly to the back of the wedge when theſe two forces balance each other: *FA* is the preſſure of the wedge upon the plane *EH*, and *DE* the preſſure of the body againſt the plane parallel to *KE* (or it is the tenſion of the ſtring *EB*) occaſioned by the action of the two former forces when they balance each other.

9. The right angled triangles *AED*, *AEF*, having one common hypothenuſe *AE*, are ſimilar to the right angled triangles *EPD*, *EPH*, having one common hypothenuſe *EP*; the ſides of the former being reſpectively perpendicular to the ſides of the latter. Therefore *DA*, *FE*, *FA* and *DE*, are reſpectively as *ED*, *PH*, *EH* and *PD*, or making *EP* radius, as the ſine of $EPD = KEP$; the ſine of *PEH*; the

coſine of *PEH*; and the coſine of *EPD*, or *KEP*; ſo that the force applied to the body in the direction *KE*, is to that on the back of the wedge which balances it; as the ſine of *KEP* the angle of direction of the force upon *E*, to the ſine of *PEH* the angle of the wedge: The preſſure on the plane *LN* is repreſented by the coſine of the latter angle, and the preſſure againſt the plane parallel to *KE*, by the coſine of the former angle.

10. *Cor.* 1. Let *PA* cut *EH* in *I*, and draw *PK* parallel to *EH* meeting *EK* in *K*, and while the wedge moves upon the plane *NL* in the direction *NL* or *PK* through the ſpace $PK = IE$, the body *E* will be raiſed in its proper direction through $EK = PI$: therefore their reſpective contemporary velocities eſtimated in the directions in which the forces on each are reſpectively applied, are as *EI* to *PI*, or becauſe of the ſimilar triangles *EDI*, *PIH*, as *ED* to *PH*, or reciprocally as thoſe forces.

11. *Cor.* 2. When *KE* is perpendicular to *PE*, the point *D* coincides with *P*, and $ED = EP$, and *PD* is nothing; that is, the preſſure againſt the ſuſtaining plane (as we called it) is nothing. The other three

forces are as *EP*, *PH*, and *HE*, the ſides of the wedge to which they are reſpectively at right angles. This is the caſe commonly conſidered by writers on mechanics.

12. *Cor.* 3. When *KE* is parallel to *PH*, the points *D* and *H* coincide, and $ED = EH$, and the force on the body in the direction *KE*, is to that on the back of the wedge which balances it, as *EH* is to *PH*. *PD* is likewiſe equal to *PH*, that is, the preſſure againſt the ſuſtaining plane is equal to the force applied to the back of the wedge.

13. *Cor.* 4. If the wedge be in the form of an iſoſceles triangle *ABC* (fig. 10) whoſe ſides are *AC* and *BC*, back *AB*, perpendicular height *CH*, and be applied to ſeparate two bodies *E* and *e* in directions *EF*, and *ef* parallel to the back but oppoſite to each other, which bodies are kept in their reſpective directions by bearing againſt a plane, or by a ſtring in the manner before laid down; then when the wedge is in equilibrio the perpendicular force on the back, will be to each of the (equal) forces *FE* or *fe* on the ſides, as the whole back *AB*, is to *CH* the perpendicular height of of the wedge.

This will appear by conſidering ſuch a wedge as is compoſed of two rectangular ones, *ACH* and *BCH*. In this caſe the preſſure of the two wedges *ACH* and *BCH* againſt one another (in the directions *AH* and *BH*) being equal and oppoſite will deſtroy each others effect. Now *FE* the force on the ſide *AC*, will be balanced by a perpendicular force on the back, which is to *FE* as *AH* is to *CH* by par. 12; and another perpendicular force on the back equal to the former will balance *fe* the force on the other ſide *BC*. Therefore the whole perpendicular force on the back, which balances both forces on the ſides, is to one only of thoſe forces as 2*AH* or *AB* is to *CH*.

SCHOLIUM I.

14. The caſe laſt deſcribed is exactly that of the machines generally made uſe of to ſhow experimentally the power of the wedge. In that deſcribed by *'s Graveſande* in the firſt and ſecond editions of his experimental philoſophy; by *Hawksbee* and *Whiſton* in their courſe of lectures; by *Deſaguliers* in his courſe of experimental philoſophy, Lect. 3. par. 58, in all theſe the rollers to be ſeparated by the wedge, reſt

upon the edge of a braſs frame which determines the direction in which they muſt move.

Theſe rollers are drawn together by weights tied to ſtrings which paſſing over a pulley run along the edge of the frame, and therefore the weights act upon the rollers in the direction of that edge. The braſs frame being ſet level, the wedge is put between the rollers, its back being parallel to the frame and to the horizon. The wedge is then loaden with weights till it begins to ſeparate the rollers, which will be when the weight of the wedge itſelf and its load, is to the weight with which each roller *ſeparately* is drawn along the frame, as the back of the wedge is to its perpendicular height.

15. In the machine deſcribed by *'s Graveſande*, in his third edition, N^{r} 279, and by Mr. *Ferguſon* in his lectures, page 68. Ed. 1760, the rollers inſtead of reſting upon the horizontal edge of a braſs frame, are ſuſpended by long lines which not only ſupport the weight of theſe rollers, but alſo determine the direction in which they muſt move when ſeparated, namely at right angles to theſe lines; that is, in a direction

parallel to the horizon as in the former machine.

16. It ſeems not to be obſerved that when the wedge is put between the rollers, the reaction of the braſs plane on which they reſt in the former machine, and the tenſion of the ſuſpending lines in the latter, is increaſed by the whole weight of the wedge and its load. This would appear very plain in the latter machine if the lines by which the rollers are ſuſpended, paſſed over pullies inſtead of being fixed to hooks in the cieling: For the rollers might then be balanced ſo as to hang ſuſpended in equilibrio before the wedge is put between them. In ſuch a caſe when the wedge is put in, that equilibrium will be deſtroyed, and to reſtore it, the rollers muſt be drawn upwards by the addition of a weight equal to the wedge and its load.

17. It therefore appears plainly that in making this experiment the rollers are not only drawn horizontally, but both of them upwards; that the rollers are each of them acted upon, not by one oblique force only, as theſe authors ſuppoſe, but by two; that the action of the wedge againſt the rollers (when they thus balance each other) does

of neceſſity create an additional tenſion of the fixed lines in *'s Graveſande's* latter experiment, and an additional preſſure upon the braſs plane in his former experiment; and laſtly, that the quantity of this additional action of the rollers upon the braſs plane and its conſequent reaction, is exactly what we determined it to be in par. 12.

18. The two forces which thus act upon each roller are both of them oblique to the ſide of the wedge but lie on oppoſite ſides of the perpendicular, their directions being at right angles to each other. Now theſe two forces will of neceſſity ASSUME ſuch a proportion that the force compounded of them both ſhall be perpendicular to the ſide of the wedge otherwiſe there could be no equilibrium. For the force compounded of them both is ſuſtained by another ſingle force, namely the reaction of the ſide of the wedge, whoſe direction is at right angles to this ſurface, and upon this principle the proportion of theſe three forces was determined in par. 6. Thus then the direction in which the rollers act againſt the wedge, is always perpendicular to its ſides however it may ſeem otherwiſe. A machine may be made in which an equilibrium ſhall

be produced when TWO forces act upon each roller in directions oblique to the ſides of the wedge; becauſe the rollers may in this caſe be urged perpendicularly againſt thoſe ſides; but no machine can be made in which *all* the parts ſhall be at reſt, when ONE force only acts on each roller in a direction oblique to the ſide of the wedge. See par. 21. and 28.

19. Whilſt we are upon this ſubject it may not be amiſs to take notice of one or two other circumſtances in theſe experiments which have occaſioned much needleſs diſputation. In all the machines but that deſcribed by *'s Graveſande* in his third edition, the ſtrings which draw the rollers together run over fixed pullies; that is, fixed to the wall of the room or to ſome firm body not connected with the rollers. Each weight then ſhows the force acting upon the roller it draws along, and the ſum of the two weights is the ſum of the two forces acting upon the rollers. In the machine deſcribed in *'s Graveſande*'s third edition, a pulley is faſtened to one of the moveable rollers, and a ſtring being tied to the other roller is put over that pulley, at the end of which is hung a ſingle weight by which

both rollers are drawn together horizontally. In this laſt caſe each roller is acted upon by the whole of that ſingle weight, and the ſum of the two forces acting upon the two rollers is twice that ſingle weight. Now what ſhall we underſtand by the force with which the rollers *cohere?* or with which they are *drawn together?* or what ſhall we underſtand by the *reſiſtance* to be overcome? ſhall we underſtand by it, the force with which each roller ſingly is drawn along the braſs frame? or ſhall we underſtand by it the ſum of the two forces by which the rollers are drawn along the braſs frame in oppoſite directions, and which is double the former force? We have ſhown in par. 13, that in the former ſenſe of the words the force on the back of the wedge, (or as it is often called, the *power)* is to the *reſiſtance*, as the whole back of the wedge is to its perpendicular height; and therefore taking *reſiſtance* in the latter ſenſe, the *power* will be to the *reſiſtance* as the whole back to twice the perpendicular height or as *half the back to the perpendicular height.* All the difficulty here is about *words*; by what names we ſhould, or rather by what names we *do* uſually call things, and if the language held on theſe occaſions be not ſettled by an uniform uſe of the terms among writers, no wonder that confuſion ſhould follow. Settle but the meaning of the terms and all diſputing vaniſhes; but dealers in this ſort of philoſophy, not ſuſpecting that they are talking to one another in a different language, *make experiments* to prove, as they ſay, the truth of their propoſitions. The experiments on both ſides always ſucceed, and to our ſurpriſe, the controverſy is no nearer to an end. —— Why? —— Becauſe theſe philoſophers are making experiments to find —— *the meaning of a word.* *

SCHOLIUM II.

20. Moſt of the writers on mechanics, in demonſtrating the power of the wedge, make uſe of the rule mentioned by Newton, "*that in mechanical engines, thoſe powers balance each other, which are reciprocally as the velocities of the parts of the engines to which they*

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* The diſpute about the FORCE OF BODIES IN MOTION, in which all the philoſophers in Europe were engaged for half a century together, was undoubtedly of this ſort. See *An eſſay on quantity* by Mr. *Reid*, in the Philoſophical Tranſactions, Nr 489, Vol. XLV, for the year 1748, or *Martyn*'s Abridgement, Vol. X. Part 1. page 22. This little eſſay, hitherto unnoticed, well deſerves the attention of every one who would underſtand either mathematical or philoſophical ſubjects *clearly*.

they are applied, eſtimated according to the reſpective directions in which the powers act." * Newton takes notice of the analogy of this rule to the rules concerning the colliſion of bodies, but gives no proof of it; nor does *Mac Laurin*, who mentions it from Newton, attempt to demonſtrate it. † Newton himſelf demonſtrates the fundamental propoſition about the lever, by the reſolution of forces; a doctrine ultimately referred by him to the firſt and ſecond laws of motion, which laws he lays down as firſt principles or general axioms from which all particular kinds of motion are to be mathematically derived. ‡ The beſt writers following Newton's ton's example, demonſtrate the other mechanical powers from the ſame principles,* but in treating of the wedge, always reſtrain their proof to a particular caſe where the forces are applied at right angles to the ſides of the wedge. It may be ſaid, perhaps, that three forces only (one to each ſide) cannot be otherwiſe applied ſo as to make an equilibrium. Let us ſee however what determination the theory gives in one of the ſimpleſt caſes of oblique forces, namely, when two equal forces act on the ſides of an iſoſceles wedge in directions parallel to the back but oppoſite to each other, and are ſuſtained by a third force acting perpendicularly on the back of the wedge.†

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* *Principia. Cor.* VI. *ad Leg. mot. pag.* 26. *Ed.* 3.

† *Mac Laurin*'s account, &c. Book. II. Chap. 3. Sect. 2.

‡ Or as we call it *ſolved*. To ſolve a phenomenon is only to derive it from, or ſhow that it is included in the general rules of motion. Thus we reduce that vaſt variety of motions we ſee in the world into a few general claſſes, and ſhow that the univerſe thus regulated by a few ſimple laws is the effect of ſupreme intelligence. To ſolve a phenomenon then, is not to aſſign its cauſe. When we ſay that gravity is the *cauſe* of the moon's motion, we mean only that its motion is of the ſame ſort with that of other bodies near the earth; that the moon's motion is conformable in its circumſtances to certain laws of motion which obtain univerſally; as to the *cauſe*, for aught we know, an angel may carry it about.

Newton ſufficiently warns his readers, (if they will take any warning) againſt this common conceit that philoſophy has to do with cauſes. See *Principia*. Def. VIII. at the end. Page 6. Ed. 3. Want of attention to this has introduced a great deal of abſurd metaphyſics into natural philoſophy, of which the reader may ſee enough in a paper by Dr. *S. Clarke*, on the force of bodies in motion. Philoſ. Tranſ. Nr 401.

21. Before we enter upon the ſolution

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* *Mac Laurin*'s account, &c. *Morgan*'s notes on *Rohault*, republiſhed at *Cambridge* 1770.

The futility of the common proof, drawn from what is called the equality of *momentums*, is ſufficiently ſhown by Dr. *Hamilton* in his ingenious eſſay on the principles of mechanics.

† Dr. *Hamilton* in his eſſay, page 165, propoſes to conſider the caſe of oblique forces, but his demonſtration, if concluſive, reſpects a very different one. In this demonſtration the *protruding* force *EP* (fig. 11) is conſidered as equivalent to two others *GP* and *PE*; that is, the perpendicular reaction of the ſide of the wedge againſt the body *E* will be ſuſtained by the joint action of the two oblique forces *GP* and *EG* upon the body: True —— but then we ſuppoſe the body *E* to be ſuſtained not by one, but by two forces, both of them (ſeparately) oblique to the ſide of the wedge, but conjointly perpendicular; both neceſſary to make an equilibrium, therefore neither can properly be ſaid to *be loſt*.

of this problem, it may be proper to premiſe an account of the nature and laws of theſe oblique forces.

Strictly ſpeaking, there is no ſuch thing in the abſtract as an oblique force, or a force producing a motion in a *ſingle body* not in the direction of that force. The motion produced in every body is ſuppoſed to be in the direction of the force that produces it. Indeed the direction of the motion produced is what marks out the direction of the force producing it, and the quantity of this motion is the meaſure of that force. But one force only may produce a motion in TWO bodies (acting againſt each other) neither of which motions *taken ſeparately* ſhall be in the direction of the original force producing them. Thus one body may be urged againſt (or may ſtrike) the ſurface of another in a direction oblique to that ſurface, and then (ſuppoſing them hard) the motion produced in each of theſe *two* bodies (conſidered ſeparately) will neither be in the direction of the original force, nor will its quantity be ſo great as what the original force alone would produce. To determine the effect of ſuch an oblique action of one body againſt another, the original force is to

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be reſolved into two, one parallel and one perpendicular to the ſurface of the body acted upon. The ſeparate effect of theſe two forces will thus be manifeſt, and by the laws of motion, the effect of two ſuch forces is the ſame as that of the original force alone. Now the quantity and direction of the motion produced in *the body acting againſt the plane* is ſuch as would be produced by the parallel force alone. The perpendicular force indeed urges the body againſt the plane (whoſe reaction ſuſtains that force) but creates no motion in this body along the plane.* Again; the quantity and direction of the motion impreſſed on *the body whoſe ſurface is acted upon*, is ſuch as would be produced by the perpendicular force alone, the parallel force creating no preſſure of the *acting body* againſt that plane. Having thus laid down the rule for finding the direction of the forces impreſſed on theſe bodies ſeparately and their proportion to the original force, we proceed to the ſolution of the problem.

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* If both theſe bodies be not hard, and their ſurface ſmooth, this parallel motion of the body along the plane will be prevented by the indenting, or by their roughneſs, that is, by the action of theſe bodies againſt each other. The original force in ſuch a caſe cannot generate a ſeparate motion in each body; both will then have one common motion juſt as if one body had preſſed or ſtruck the ſurface of the other perpendicularly.

22. Let *ABC* (fig. 10) be the iſoſceles wedge, whoſe angular point is *C*, ſides *AC* and *BC*, back *AB*, perpendicular height *HC*: Let *FE* repreſent the quantity and direction of the force applied to one of the ſides; reſolve this into two other forces, *FD* and *DE*, the former parallel, the latter perpendicular to the ſide *AC*, and the original oblique force *FE* will have juſt the ſame effect upon the wedge as a leſſer perpendicular force *DE*, the former being to the latter as *AC* is to *HC*. But this laſt perpendicular force on the ſide *AC* is to that on the back which balances it, as *AC* is to *AH*, by par. 11; whence compounding theſe ratios, the oblique force againſt one ſide of the wedge is to the perpendicular force on the back which balances it, as AC^2 is to $AH \times HC$.

The oblique force *fe* on the other ſide of the wedge being equal to *FE*, will require another perpendicular force on the back to balance it, equal to the former perpendicular force; whence the whole force on both ſides of the wedge, is to the whole force on the back, as AC^2 is to $AH \times HC$, or as the ſquare of the ſide of the wedge to the rectangle under half the back and the perpendicular height.

23. There is no paralogiſm in this demonſtration, and the concluſion would obtain in fact, if a machine could be conſtructed in which all the circumſtances ſhould exactly agree to thoſe here ſuppoſed, that is, to the *data* of the problem. If for inſtance, two rollers could be made to preſs the ſides of the wedge in directions parallel to the back, but oppoſite to each other, and at the ſame time be permitted to roll down the ſides of the wedge with part of their original force; —— but nothing like this is the caſe of the machines before mentioned, though intended to ſhow experimentally the power of the wedge in the circumſtances laid down in this problem. Therefore the experiments made with thoſe machines, though they undoubtedly ſucceed juſt as related, yet do by no means prove the propoſition they are brought to prove, nor do they invalidate the reaſoning here advanced. On the other hand, the problem, as it is here propoſed, perhaps will not ſuit any caſe in which the wedge can be *practically* introduced, but muſt ever remain a matter of *uſeleſs ſpeculation.*

APPENDIX.

What follows is a deſcription of two machines intended to ſhow the power of the wedge in the caſe before mentioned. How far their circumſtances anſwer to the *data* of the problem in par. 20, is left to the reader's conſideration.

24. Let a roller *F*, (fig. 11) be drawn by a ſtring *DF* running over a pulley *D*, ſo fixed, that when the roller bears againſt the ſide *BC* of the iſoſceles wedge *ABC*, the ſtring *DF* ſhall be parallel to *BC*. At the ſame time let this roller be drawn in the direction *FE* by another ſtring *FE* running parallel to *AC* the back of the wedge. Let another roller *E* be applied in like manner to *AB* the other ſide of the wedge. Laſtly, let the force with which the rollers act againſt the ſides, be balanced by another force *P* acting perpendicularly to *AC* the back of the wedge: then will the ſum of the two forces on the ſides be to the force on the back, as AB^2 is to $AP \times PB$.

The rollers are here ſuppoſed to have no weight of their own, but to owe their whole force to weights that draw them.

25. The roller *F* is here acted upon by

two forces, one of which being parallel to *BC*, oppoſes the motion of the roller along the ſide *C B*, but adds nothing to its preſſure againſt the ſide of the wedge. The roller is alſo drawn in the direction *F E*, oblique to the ſide *BC*, but it can move only in the direction *FP*, perpendicular to the ſtring *F D*, and therefore perpendicular to the ſide of the wedge.

26. Another machine better adapted to practice is deſcribed by Mr. *Ferguſon* in his lectures, page 67. Ed. 1760. Plate VI. Fig. 6. and is here delineated in Fig. 12. In this machine the wedge *ABC* is not iſoſceles but rectangular. The ſtring *FG* (which muſt be always kept parallel to *B C*) may be either fixed to an hook, or made to run over a pulley *H*, in which caſe the cylinder *F* muſt be poiſed by a weight *W*. The equilibrium will be when the weight of the cylinder *F* is to the weight *K*, that draws the wedge along the horizontal plane (in the direction *A I*, perpendicular to the back *AB*) as BC^2 is to $AB \times AC$; as will be ſhown preſently in par. 29.

27. The cylinder *F* is here acted upon by two forces, the weight *W* and the force of gravity; the former only oppoſes the de-

deſcent of the cylinder down *BC* the face of the wedge, and creates no preſſure of the cylinder againſt that face. The force of gravity acts in a direction parallel to the back *BA*, and therefore oblique to the face *BC*; yet while the wedge is drawn along the horizontal plane, the cylinder *F* conſtantly riſes in a line which is (relatively to fixed objects) perpendicular to the face *BC*, and can move in no other direction.

28. If the cylinder was ſuffered to deſcend freely down the face of the wedge *BC*, and the wedge was kept from running backwards (during the time of deſcent) by the weight *K*; the caſe would then exactly anſwer the *data* of the problem, in par. 20, excepting that there would be here one oblique force only; the other two being perpendicular to the ſides of the wedge on which they act.

29. From *A* let fall *AD* perpendicular to *BC*, and from *D* let fall *DE* perpendicular to *AC*. Let now *BA* repreſent the weight of the cylinder, then the force (of gravity) *BA* may be reſolved into the forces *BD* and *DA*: Again; the force *DA* may be reſolved into the forces *DE* and *EA*. Now *BA*, *BD*, *DE* and *EA*, are to each

other as BC^2, $AB \times BC$, AC^2, and $AB \times AC$ reſpectively: therefore if BC^2 repreſents the weight of the cylinder, then $AB \times BC$ will repreſent the force with which it deſcends down the face of the wedge; AC^2 will be the preſſure of the wedge upon the horizontal plane, and $AB \times AC$ the force impelling the wedge (backwards) along the horizontal plane.—This laſt force will be greateſt when AB and AC are equal, in which caſe it will be half the weight of the cylinder.

30. Let the ſtring be fixed at H, and let the wedge be drawn under the cylinder; then when the wedge has been drawn its whole length AC on the horizontal plane, the cylinder F will be raiſed along the face of the wedge from C to D, whoſe perpendicular height, above the horizontal plane AC, is DE; therefore the cotemporary velocities of the weights F and K eſtimated in the directions in which they act, are reciprocally as AC to DE, that is, in a ratio compounded of AC to AD, and AD to DE; or as BC^2 to $AB \times AC$, that is, reciprocally as thoſe weights when they balance each other.

THE END.

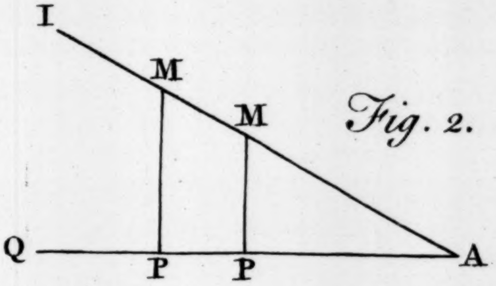

Fig. 2.

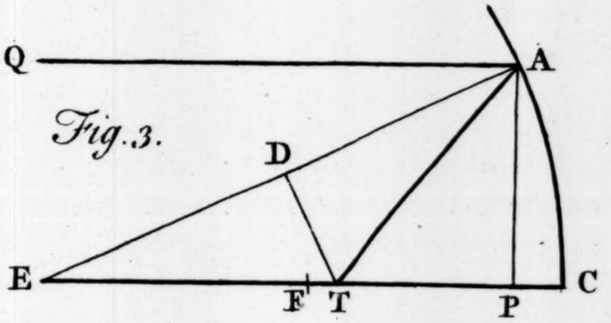

Fig. 3.

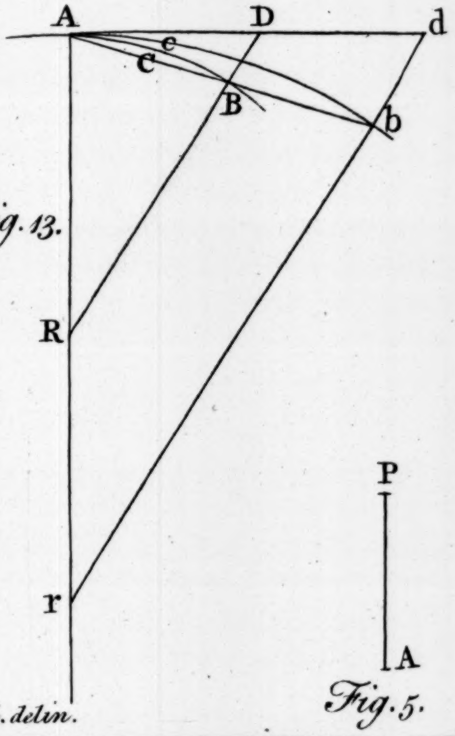

Fig. 13.

Fig. 5.

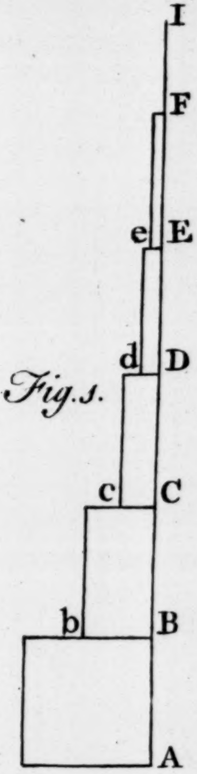

Fig. 1.

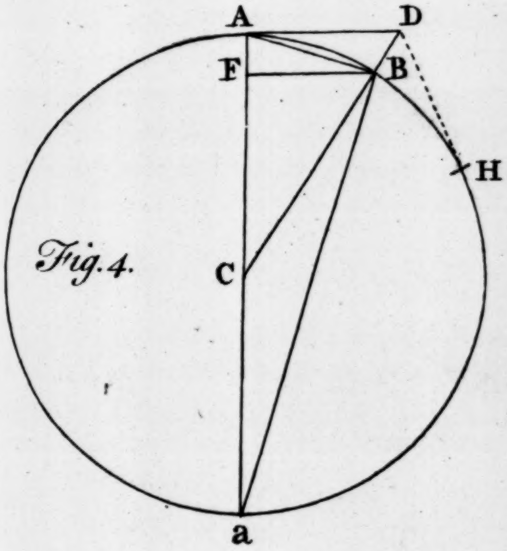

Fig. 4.

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Fig. 12.

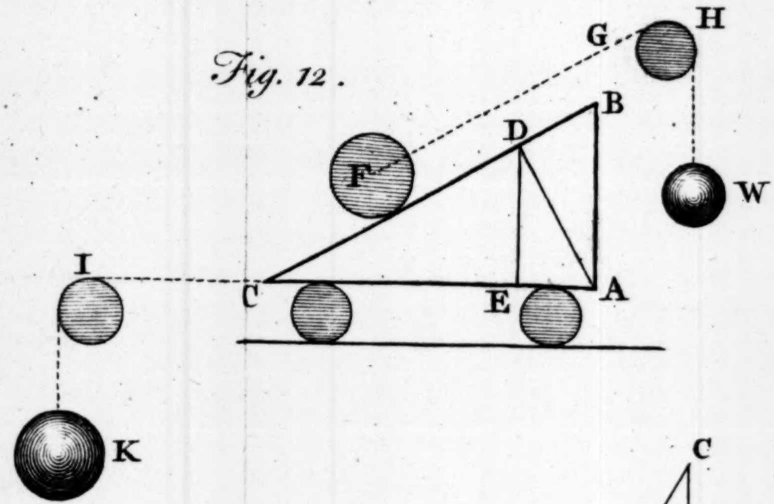


Fig. 9.

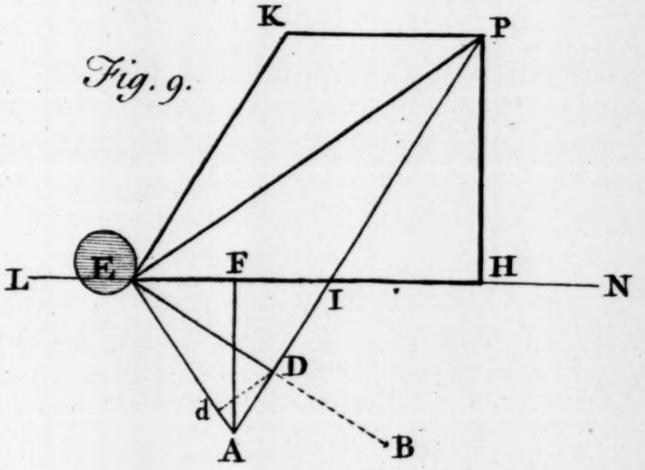


Fig. 6.

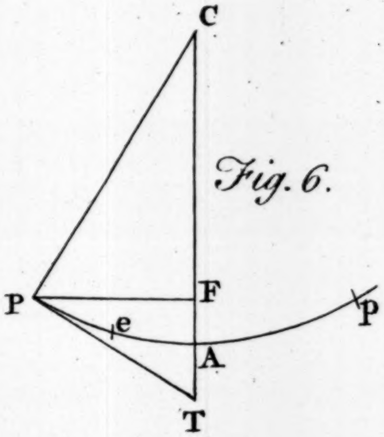


Fig. 10.

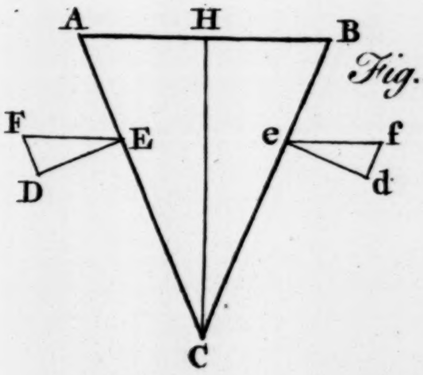


Fig. 7.

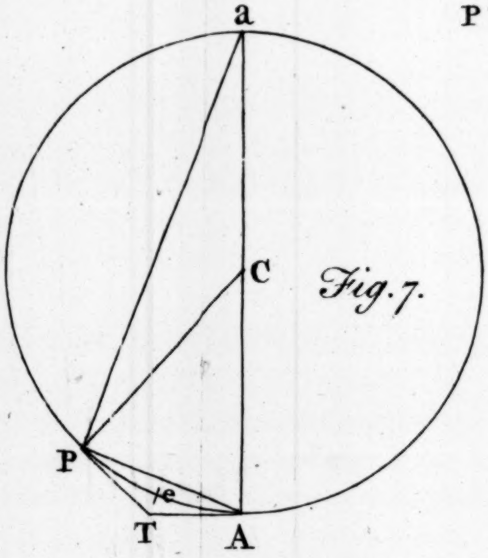


Fig. 8.

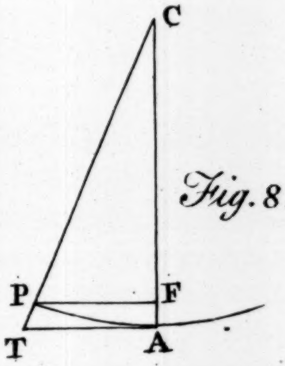
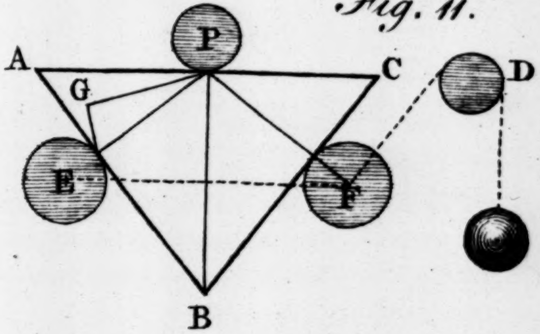


Fig. 11.



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